Complex Numbers

Complex numbers sneaked into history in a very strange way. If you want to know more about it, the last chapter in Dunham's book is warmly recommended. I'll give here a summary of the ideas. My goal is to give you a feel for how they got discovered. So you have to go back in memory to the times where you only knew real numbers.

Now the first thing to say is how they did NOT get discovered; namely by solving quadratic equations. Consider the equation

$$x^2 + 1 = 0 \ .$$

If somebody asks to find the solutions then a perfectly reasonable answer is, that there isn't any. Just draw a graph for this function; it never hits the x-axis.

Problem

Find the condition that the equation

$$x^2 + bx + c = 0$$

has no solution.

Mind you that people toyed around with the idea that

$$\sqrt{-1} \text{ and } -\sqrt{-1}$$

solves the equation $x^2+1 = 0$ but they could not do anything with it. What is the meaning of $\sqrt{-1}$? Certainly that its square is -1. Now this is of course wonderful since we can count with completely new numbers and order, e.g., $\sqrt{-1}$ cappucino. From the looks of the waiter you will discern that this is not a good idea; it does not make sense. Quantities in real life are real numbers and hopefully even simpler numbers, namely rational ones. You see, from this perspective $\sqrt{2}$ is not more real either. In short, nobody in his right mind would introduce a symbol just to solve the silly equation $x^2 + 1$ for its own sake. This what happened historically. People simply said that quadratic equations sometimes have solutions and sometimes not.

I would like to explain a problem where complex numbers are really useful and were actually discovered, and this is the problem of solving cubic equations. Complex numbers come to our aid in an unexpected way.

Consider the cubic equation

$$x^3 + bx = c \ . \tag{1}$$

The solution of this equation is a tricky problem but the solution can be found in a number of cases quite easily by guessing. In fact there is always a real solution .

Problem

Show that every cubic equation has at least one real solution.

Hint

Take equation (1) and decide what happens when x gets very large positive and very large negative.

Solution

As x gets very large the left side of equation (1) gets very large positive in particular it will eventually exceed the value c. When x gets very large negative the left side of the equation does the same and will eventually take a value that is below c. Since the function $x^3 + bx$ is continuous it must hit the value c when moving x from large positive values to large negative values.

Knowing that a solution exists does not really tell us how to get at it. Fortunately the Italian mathematician Cardano (1501-1576) managed to give a formula that will do the job, or so he thought. Here it is

$$x = \sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} \; .$$

It is natural to wonder how Cardano came up with that formula but we do not concern ourselves with this very interesting question but instead just check it.

Problem

Check that Cardano's formula indeed solves equation (1).

Hint

Remember from the binomial formula that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \; .$$

Solution

Set

$$a = \sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}$$

and

$$b = \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}$$

and compute $(a+b)^3$. Using the Binomial formula we get that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$= \frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}} - 3\left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}\right)^{2/3}\left(-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}\right)^{1/3}$$

$$+3\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3}\left(-\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{2/3}-\left(-\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right) . \qquad (2)$$

The first and the last term simplify to yield c, which is good news. The second term can be written as

$$-3\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3}\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3}\left(-\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3} ,$$

$$= -3\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3}\left[\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)\left(-\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)\right]^{1/3}$$

$$= -3\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3}\frac{b}{3} .$$

Similarly the first term in (2) can be written as

$$+3\left(-\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3}\frac{b}{3} .$$

Collecting the terms we get that

$$x^3 = c - 3\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3}\frac{b}{3} + 3\left(-\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right)^{1/3}\frac{b}{3}$$

which is nothing else than

$$c - bx ,$$

what we had to show.

The formula seems to work! Do you see a problem with the above computation? If not consider the following nice example, taken from Dunham's book.

Problem

Find a solution of the equation

$$x^3 + 24x = 56 .$$

Solution

We have that $c = 56$ and $b = 24$. Therefore

$$\sqrt{\frac{c^2}{4}+\frac{b^3}{27}} = \sqrt{784+512} = \sqrt{1296} = 36 .$$

Hence

$$x = (28 + 36)^{1/3} - (-28 + 36)^{1/3} = 2 \ .$$

Thus, we have 2 as one solution.

You might wonder why we consider somewhat artificial equations of the form

$$x^3 + bx = c \ ? \tag{3}$$

The full cubic equation looks rather like

$$x^3 + ax^2 + bx = c \ . \tag{4}$$

The reason is that if we set $x = y - a/3$ we get the equation

$$y^3 + (b - a^2/3)y = (c - \frac{2a^3}{27} + \frac{ab}{3}) \ ,$$

which is precisely of the form for which Cardano's formula works.

So we are in very good shape. We can bring every cubic equation into Cardano's form and then apply his formula to get one root. By a long division this root can be factored from the cubic equation and we are left with a quadratic equation which can be solved to get the other roots if there are any.

Here is another example taken from Dunham's book.

Problem

Solve the equation

$$x^3 - 78x = 220 \ .$$

Solution

In our example $b = -78$ and $c = 220$. Next we compute

$$\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3 = 110^2 - 26^3 = 12,100 - 17,576 = -5476 \ .$$

We have to take the square root of this last number but this cannot be done since this number is negative. Please do not forget that you are not supposed to know anything about complex numbers!

What do we conclude from the previous problem? That Cardano's formula is not working? Well, it seems that way since the equation of the previous problem has three real roots! They are

$$10, \ -5 + \sqrt{3}, \ 5 + \sqrt{3} \ .$$

This is really bad news. There are three real roots and Cardano's formula does not even give one of them!

What is also disconcerting is that we made a mistake when we checked Cardano's formula. What we did was to ignore the fact that we took the root of a number that might be negative depending on the coefficients of the equation.

Progress was made by the Italian mathematician R. Bombelli (1526-1572). He decided to compute with the root of negative numbers. He assumed all the rules of multiplication and added the additional rule that

$$\sqrt{-1}\sqrt{-1} = -1 \ .$$

This is a purely formal convention, since nobody knows what $\sqrt{-1}$ really is. Again, let me add that we behave as if we know a lot of things, like $\sqrt{2}$ but if we are really honest we do not know much about this number either, except that its square is 2 and we can compute it in terms of decimal approximations.

But back to the number $\sqrt{-1}$ and to Bombelli. He did the following computation

$$\sqrt{-5476} = \sqrt{5476}\sqrt{-1} \ .$$

A little bit of thinking or pushing buttons shows that

$$\sqrt{5476} = 74$$

and hence we get from Cardano's formula that

$$x =^3 \sqrt{110 + 74\sqrt{-1}} -^3 \sqrt{-110 + 74\sqrt{-1}} \ .$$

This is nice but what is it good for. Now you have to remember that you should compute with $\sqrt{-1}$ as you would with an ordinary number. So lets try to compute the cubic root of the number $-110 + 74\sqrt{-1}$. This we do by guessing. Start with the number

$$-5 + \sqrt{-1} \ ,$$

and cube it. You know how to do that. Simply apply the binomial formula. Therefore

$$\left(-5 + \sqrt{-1}\right)^3 = (-5)^3 + 3(-5)^2\sqrt{-1} + 3(-5)\sqrt{-1}^2 + \sqrt{-1}^3 \ .$$

Now remember that $\sqrt{-1}\sqrt{-1} = -1$ and you get

$$\left(-5 + \sqrt{-1}\right)^3 = -125 + 75\sqrt{-1} + 15 - \sqrt{-1} = -110 + (75\sqrt{-1} - \sqrt{-1}) = -110 + 74\sqrt{-1}$$

after factoring out $\sqrt{-1}$. In a similar way you compute that

$$\left(5 + \sqrt{-1}\right)^3 = 110 + 74\sqrt{-1} \ .$$

(We were really lucky here that William Dunham did all the work for us by cooking up an example that gives simple numbers.)

That yields for our solution

$$x =^3 \sqrt{110 + 74\sqrt{-1}} -^3 \sqrt{-110 + 74\sqrt{-1}} = (5 + \sqrt{-1}) - (-5 + \sqrt{-1}) = 10 \ ,$$

since the $\sqrt{-1}$ cancel. This is one of the roots! Bombelli discovered here something amazing. He showed that Cardano's formula does make sense. You can find *real* solutions of a cubic equation but you have to go *outside* the realm of the real numbers. It is absolutely essential that you use these new numbers.

Problem

Compute

$$\left(\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}\sqrt{-1}\right)^3 \tag{5}$$

and

$$\left(-\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}\sqrt{-1}\right)^3 \ . \tag{6}$$

Using this information together with Cardano's formula compute a solution of the equation

$$x^3 - 6x = 4 \ .$$

Find all the solutions of this equation.

Solution

Using the binomial formula we get for (5) $2 + 2\sqrt{-1}$ and for (6) $-2 + 2\sqrt{-1}$. Using Cardano's formula we get as a solution

$$x = (2 + 2\sqrt{-1})^{1/3} - (-2 + 2\sqrt{-1})^{1/3}$$

which thus is reduced to

$$x = \sqrt{3} + 1 \ .$$

Next we factor our polynomial by this root

$$x^3 - 6x - 4 : (x - \sqrt{3} - 1) = x^2 + (\sqrt{3} + 1)x + 2(\sqrt{3} - 1) \ .$$

This quadratic equation can easily be solved and yields the additional real solutions

$$x = -\frac{\sqrt{3}+1}{2} \pm \frac{\sqrt{3}}{2}\sqrt{4 - 2\sqrt{3}} \ .$$

Bombelli wrote a book about his discovery; it is his third volume on algebra. In this book he took an approach to these new numbers very much as we do it today. He did not have a full theory yet, there are certain very important formulas missing, but he opened the door to a wonderful new set of numbers which we nowadays call the complex numbers. They are useful in almost all of the applied sciences. You can happily compute with them as you do with the real numbers. The following lessons should convince you of that. There is only one thing you have to remember. When you go back to the real world, e.g., when you pay for your cappucino, they don't take $5 + \sqrt{-1}$ dollar bills.

Sources

I have copied some of the examples from the wonderful book of William Dunham, entitled 'The mathematical universe', published by Wiley in 1994. One does not need a strong mathematics background to read this book but one can learn a lot of good mathematics from it and enjoy humorous stories about mathematicians.

Some of the historical facts I gleaned from the great book of André Weil, one of the great mathematician of this century: 'Number Theory, an approach through history' published by Birkhäuser in 1983. This book is not so elementary as the one above but there are many parts that can be read by anyone who is seriously interested in the field of number theory.

With best regards

Michael Loss