The Secret Collusion of Fourier and Hermite

Recall that for suitable functions f, we have established that

$$\widehat{xf}(\xi) = -\frac{1}{2\pi i}\frac{d}{d\xi}\hat{f}(\xi) \quad \text{and} \quad \widehat{\frac{d}{dx}f}(\xi) = 2\pi i\xi\hat{f}(\xi) \tag{1}$$

(For example, it suffices to assume that $f(x)$, $xf(x)$ and $f'(x)$ are all continuous and bounded in magnitude by $\frac{c}{1+x^2}$.) We have also seen that $\widehat{e^{-\pi x^2}} = e^{-\pi\xi^2}$, so if we write the Fourier transform as an operator F, we see that $e^{-\pi x^2}$ is an eigenfunction of $\mathcal{F}$ with eigenvalue 1.

Something special happens for the operators $a_{\pm}^{(x)} := \left(2\pi x \mp \frac{d}{dx}\right)$ for we calculate

$$\widehat{a_{\pm}^{(x)}f} = \left(i\frac{d}{d\xi} \mp 2\pi i\xi\right)\hat{f}(\xi) = \mp i\left(2\pi\xi \mp \frac{d}{d\xi}\right)\hat{f}(\xi) = \mp i a_{\pm}^{(\xi)}\hat{f}(\xi) \tag{2}$$

Moreover, we can calculate

$$\begin{aligned} a_{+}^{(x)}x^n e^{-\pi x^2} &= (4\pi x^{n+1} - nx^{n-1})e^{-4\pi x^2} \\ a_{-}^{(x)}x^n e^{-\pi x^2} &= nx^{n+1}e^{-4\pi x^2} \end{aligned}$$

So, inductively, if the polynomials h_n are defined by $h_0 = 1$, $(a_{+}^{(x)})^n e^{-\pi x^2} = h_n(x)e^{-\pi x^2}$, we conclude

$$\begin{aligned} h_n \text{ is a polynomial of degree } n, \text{ even if } n \text{ is even} \\ \text{odd if } n \text{ is odd.} \end{aligned} \tag{3}$$

A fact from functional analysis. Every square integrable function on $\mathbb{R}$ can be approximated (in the L^2, or r.m.s sense) by polynomials $\times e^{-\pi x^2}$.

If we designate the square integrable functions by $L^2(\mathbb{R})$, and note that this space of functions has an inner product $\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$, and a norm $\|f\|_2^2 = \langle f, f\rangle$, we can express this "completeness" lemma as:

For all $f \in L^2$, and all $\epsilon \geq 0$, there exist $N, c_0, \ldots, c_N$, such that

$$\left\| f - \sum_{n=0}^{N} c_n h_n(x)e^{-\pi x^2} \right\|_2 < \epsilon.$$

Actually, we can be more precise, because $\{h_n(x)e^{-\pi x^2}\}$ is an orthogonal basis for L^2, i.e.,

$$\langle h_n(x)e^{-\pi x^2}, h_m(x)e^{-\pi x^2}\rangle = 0 \quad \text{unless } m = n \tag{4}$$

There are various ways to prove (4). For example, you can show inductively that if $H = -\frac{d^2}{dx^2} + 4\pi^2 x^2$, then

$$Hh_n(x)e^{-\pi x^2} = (a_+a_- + 2\pi)h_n(x)e^{-\pi x^2} = 2\pi(2n+1)h_n e^{-\pi x^2}$$

so h_n and h_m with $m \neq n$ are eigenfunctions of the self-adjoint operator H with different eigenvalues, and this implies they are orthogonal. (Hint: $Hh_n(x)e^{-\pi x^2}$ also $= (a_-a_+ - 2\pi)h_n e^{-\pi x^2}$.)

Because of (2), we have

$$\widehat{h_n(x)e^{-\pi x^2}} = (-i)^n h_n(\xi)e^{-\pi\xi^2} \tag{5}$$

So $h_n(x)e^{-\pi x^2}$ is an eigenfunction of $\mathcal{F}$ with an eigenvalue of modulus 1.

The functions $h_n(x)$ are known as the Hermite polynomials. (Normalizations differ.)

Exercises

1. Given (1)–(5), show that for any $f \in L^2$

$$f(x) = \sum_{k=0}^{\infty} c_k h_k(x) e^{-\pi x^2}, \tag{6}$$

   and find the formula for c_k in terms of f and h_k.

   Note: If you have had functional analysis, interpret (6) in the L^2 sense. If not, assume f is not only square-integrable but also continuous and decays sufficiently rapidly at ∞ to guarantee convergence of any integrals you wish to do.)

2. Show that for $f, g \in L^2(\mathbb{R})$,

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle. \tag{7}$$

   In particular, $\|f\|_2 = \|\hat{f}\|_2$.

3. Prove the inversion formula for the Fourier transform for $f \in L^2$, using (1)–(7) and show that $\mathcal{F}$ is one-to-one and onto as a map $L^2 \to L^2$.

   (We regard f as equal to g if $\|f - g\|_2 = 0 \Leftrightarrow f = g$ a.e. Also, a bounded linear transformation on a dense subset of L^2 is uniquely defined by continuity on all of L^2.)