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Isn't there something odd about $z\cos(\frac{a}{z})$?

The Logarithm

We might define $\ln(z) = \int_1^z \frac{1}{t}\,dt$ for "suitable paths." If this works, $\frac{d}{dz}\ln(z) = \frac{1}{z}$. The fn $\frac{1}{z}$ is holo $(\mathbb{C}-\{0\})$, but this set is not simply connected. We can correct this with an artificial bdry, which we call a "branch cut."

Define $\Omega := \mathbb{C} - (-\infty, 0]$. We want to do this because $\ln(z)$ as defined above is not single-valued on the punctured plane:

Let z vary along the unit circle CCW beginning at $z=1$. When z returns to $+1$, $\ln(z)$ becomes $2\pi i$, which is contrary to starting with $\ln(1)=0$.

Does $\ln(z)$ behave like the good, old real logarithm?

Fix w and consider $z \to \ln(zw)$ where we assume $\arg(z), \arg(w), \arg(zw) = \arg(z)+\arg(w) \in (-\pi, \pi)$.

Then $\frac{d}{dz}\ln(zw) = w \frac{d}{d\zeta}\ln\zeta \Big|_{zw=\zeta} = \frac{w}{zw} = \frac{1}{z}$

and thus $\frac{d}{dz}\left(\ln(zw) - \ln(z)\right) = 0$. Hence, under these restrictions on the argument, $\ln(zw) = \ln(z) + C_w$.

If we take $z=1$, we see that $C_w = \ln(w)$, and thus $\ln(zw) = \ln(z) + \ln(w)$.

Principle of the Argument

Morally applies to $\ln(f)$ which has $\frac{d}{dz}(\ln(f)) = \frac{f'}{f}$.

The left side is problematic because of the branch cut.

The right side is completely reasonable for meromorphic functions.

$\longrightarrow$

Suppose f is meromorphic on some disc $\mathcal{D}$ and C is a circle (ccw-oriented) in $\mathcal{D}$ s.t. C does not meet any zeros or poles of f.

$f(z)$ = principal part + regular part.

At a pole z_0 of order m, $f(z) = (z-z_0)^{-m} g(z)$ where $g(z)$ is a nonvanishing holomorphic function in a neighborhood of z_0. Then

$$\frac{f'(z)}{f(z)} = \frac{-m(z-z_0)^{-m-1} g(z) + (z-z_0)^{-m} g'(z)}{(z-z_0)^{-m} g}$$

$$= \frac{-m}{z-z_0} + \frac{g'}{g}.$$

Hence, $\frac{f'}{f}$ has a a simple pole at the position of any pole of f. Furthermore, $\mathrm{Res}_{z_0}\left(\frac{f'}{f}\right) = -m$.

Suppose now that z_1 is a zero (hence isolated) of f.

Then $f(z) = (z-z_1)^n h(z)$, $h(z)$ non vanishing & holo fn on nbhd of z_1.

and $$\frac{f'(z)}{f(z)} = \frac{n(z-z_1)^{n-1} h + (z-z_1)^n h'}{(z-z_1)^n h} = \frac{n}{z-z_1} + \frac{h'}{h}$$

~~where~~ and $\frac{h'}{h}$ is regular on nbhd of z_1.

Thus, f'/f also has a simple pole at position of any zero of f with $\mathrm{Res}_{z_1}(f'/f) = +n$.

Thus, we have proved:

<u>Thm</u>: Let f be meromorphic on $\mathcal{D} \supseteq C$, a CCW circle s.t. f is holo on nbhd of C. Then

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i\,(Z_C(f) - P_C(f))$$

where $Z_C(f)$ = # of zeroes (with multiplicities) enclosed by C.
$P_C(f)$ = " poles " " "

<u>Rouché</u>:
Suppose f is meromorphic and g is badly-behaved. Then if $|g| < |f|$ on C, we have

$$(Z_c - P_c)(f(z) + g(z)) = (Z_c - P_c)(f).$$

<u>Improved Rouché Theorem</u> (Glicksberg 1976)

Suppose f and g are meromorphic on $\mathcal{D}$. Assume f has no poles or zeroes on $C \subset \mathcal{D}$. On C, suppose $|f| > |g| - |f+g|$. Then

$$(Z_C - P_c)(f) = (Z_C - P_c)(f+g).$$

<u>Proof</u>: Observe that under the hypotheses of the theorem, $\left|\frac{g}{f+g}\right| < \left|\frac{f}{f+g}\right| + 1$ on C. Equivalently, we can write $\left|\frac{-f}{f+g} + 1\right| < \left|\frac{-f}{f+g}\right| + 1$. Thus $\frac{f}{f+g}$ is never negative (or zero). Therefore, $\ln\left(\frac{f}{f+g}\right)$ is holo on a nbhd of C. Thus $\ln\left(\frac{f}{f+g}\right)$ is a primitive for its derivative on a nbhd of C. We compute:

$$\frac{d}{dz}\ln\left(\frac{f}{f+g}\right) = \left(\frac{f}{f+g}\right)' \Big/ \left(\frac{f}{f+g}\right) = \frac{f'}{f} - \frac{(f+g)'}{(f+g)}.$$

C is a circle, so beg pt and end pt are same

$$\text{Thus } \underbrace{\int_C \left(\frac{f'}{f} - \frac{(f+g)'}{(f+g)}\right) dz}_{2\pi i\left((Z_C - P_C)(f) - (Z_c - P_c)(f+g)\right)} = \ln\left(\frac{f}{f+g}\right)_{\text{end pt}} - \ln\left(\frac{f}{f+g}\right)_{\text{beg pt}} = 0$$

$\rightarrow$

Analytic Open Mapping Theorem

Assume that $f \in holo(\Omega)$, Ω open and connected, and that $\mathcal{O} \subseteq \Omega$ is open in Ω. Then $f(\mathcal{O})$ is also open.

Pf: Suppose $w_0 \in f(\mathcal{O})$. Then for w near w_0,

$$f(z) - w = f(z) - w_0 + (w_0 - w).$$

We may assume that $w_0 = 0$, and then we have

$f_w(z) + g_w(z)$

where $g_w(z)$ is constant as far as z is concerned

Choose $\delta > 0$ to be small so that $f(z) \neq 0$ on a circle of radius δ centered at z_0 s.t. $f(z_0) = 0$. This is possible because the zeroes are isolated.

If $|w| < \varepsilon$ is sufficiently small that $|w| < \min_{\{|z-z_0|<\delta\}} |f|$. Now apply Rouché's theorem, and we conclude that the number of zeroes of $f(z) - w$ = mult of soln of $f(z) = 0$