1 Problem 1

Let $u \colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ with $u \in C^1$ satisfy

$$u_t + b \cdot Du + cu = 0 \tag{1}$$
$$u(x, 0) = g(x) \tag{2}$$

Find u.

Inspired by Evans' solution to the non-homogenous transport equation, begin by defining a function $\Phi_{x,t}(s) := u(x + bs, t + s)$ for each $x \in \mathbb{R}^n$ and each $t \in \mathbb{R}^+$.

Differentiating with respect to s, we find $\Phi'_{x,t}(s) = Du(x + bs, t + s) \cdot b + u_t$. This allows us to rewrite (1) as follows.

$$\Phi'_{x,t}(s) + c\Phi_{x,t}(s) = 0$$

This is an ODE for fixed x and t. The solution will then be a function of x and t:

$$\Phi_{x,t}(s) = A_{x,t}e^{-cs}$$

or

$$u(x + bs, t + s) = A_{x,t}e^{-cs} \tag{3}$$

Evaluate (3) at s=-t and use (2) to obtain

$$u(x - bt, 0) = A_{x,t}e^{ct} = g(x - bt)$$
$$A_{x,t} = g(x - bt)e^{-ct}$$

We now see (3) can be written

$$u(x + bs, t + s) = g(x - bt)e^{-ct}e^{-cs}$$

Substitute $x' = x + bs$ and $t' = t + s$ to write

$$u(x', t') = g(x' - bs - bt)e^{-c(t+s)} = g(x' - bt')e^{-ct'}$$

Now it'd be a good idea to check that this satisfies the equations so assume u is defined by the derived equation. Evaluating at $t' = 0$ this clearly satisfies (2). To check that (1) is true evaluate u_t as follows.

$$u_t(x, t) = -ce^{-ct}g(x - bt) + -e^{-ct}b \cdot Dg(x - bt) = -cu - b \cdot Du(x, t)$$

2 Problem 2

Let $u \in C^2(\overline{U})$. If $-\triangle u \leq 0$ in U we say u is subharmonic.