2.1 Mean value property

This proof is almost identical to the mean value theorem proof given for harmonic functions. First define

$$\Phi(r) := ⨍_{\partial B(x,r)} u(y)dS(y)$$

Substituting $y = x + rz$ gives $r^{n-1}dS(z) = dS(y)$ and

$$\Phi(r) = ⨍_{\partial B(0,1)} u(x+rz)dS(z)$$

Next we take the derivative of Φ respect r and justify bringing the differentiation into the integral sign. Using the mean value theorem for one variable we have for some ξ between 0 and h

$$\begin{aligned}\Phi'(r) &= \lim_{h\to 0} ⨍_{\partial B(0,1)} \frac{u(x+(r+h)z) - u(x+rz)}{h} dS(z) \\ &= \lim_{h\to 0} ⨍_{\partial B(0,1)} u_r(x+(r+\xi)z)dS(z)\end{aligned}$$

Since the integrand for small enough h is entirely interior to $\overline{U}$, it is continuous on a compact set, and therefore uniformly continuous. To show that the sequence above converges to the integral of the derivative, consider the difference below. Pick $\epsilon > 0$. There must exist some $\delta > 0$ such that

$$\begin{aligned}|u_r(x+(r+h)z) - u_r(x+rz)| < \epsilon \\ \forall |h| < \delta\end{aligned}$$

Choosing such an h we have

$$\begin{aligned}&|⨍_{\partial B(0,1)} u_r(x+(r+\xi)z) - u_r(x+rz)dS(z)| \\ \le &⨍_{\partial B(0,1)} |u_r(x+(r+\xi)z) - u_r(x+rz)|dS(z) \\ &\le ⨍_{\partial B(0,1)} \epsilon dS(z) = \epsilon\end{aligned}$$

Letting $h \to 0$ lets us bring the differentiation inside just like we wanted to. Next use the chain rule to rewrite u_r and return to the original variable y.

$$\begin{aligned}\Phi'(r) &= ⨍_{\partial B(x,r)} z \cdot Du(y)dS(y) \\ = ⨍_{\partial B(x,r)} \frac{y-x}{r} \cdot Du(y)dS(y) &= ⨍_{\partial B(x,r)} \nu \cdot Du(y)dS(y)\end{aligned}$$

Using Green's theorem this can be written

$$\frac{1}{\alpha(n)nr^{n-1}}\int_{\partial B(x,r)} \nu\cdot Du(y)dS(y)$$
$$= \frac{r}{\alpha(n)r^n}\int_{B(x,r)} \triangle u dy$$
$$\geq 0$$

This means $\Phi(r)$ is increasing for all $r > 0$. A lower bound on $\Phi(r)$ could therefore be established by taking the limit as $r \to 0$. This limit is:

$$\lim_{r\to 0}\Phi(r) =$$
$$\lim_{r\to 0} ⨍_{\partial B(x,r)} u(y)dS(y) = u(x)$$

Therefore $u(x) \leq ⨍_{\partial B(x,r)} u(y)dS(y)$ Rewrite this as

$$\alpha(n)nr^{n-1}u(x) = \int_{\partial B(x,r)} u(y)dS(y)$$

Integrating both sides with respect to r gives:

$$\alpha(n)r^n u(x) \leq \int_0^r \int_{\partial B(x,r')} u(y)dS(y)dr'$$
$$= \int_{B(x,r)} u(y)dy \Rightarrow u(x) \leq ⨍_{B(x,r)} u(y)dy$$

2.2 Maximum principle

The claim is $\max_{\overline{U}} u = \max_{\partial U} u$. A theorem similar to this is shown in the book for harmonic functions but despite the fact that it states connectedness of U is unnecessary, it actually seems to use it since it proves (ii) then says (i) follows from (ii) without proof. Anyway the argument here is a bit different so connectedness never comes up. First suppose the maximum M is obtained for some $x_0 \in U$. Using the mean value property yields

$$u(x_0) = M \leq ⨍_{B(x,r)} u(y)dy$$

A contradiction could be obtained if it were shown that the right side of this equation is actually strictly less than M. This is done as follows. Suppose $u(y) < M$ on some set of positive measure.

$$⨍_{B(x,r)} u(y)dy =$$
$$\frac{1}{\alpha(n)r^n}\int_{\{y:\, u(y)=M\}} u(y)dy + \frac{1}{\alpha(n)r^n}\int_{\{y:\, u(y)<M\}} u(y)dy$$
$$< \frac{1}{\alpha(n)r^n}\int_{\{y:\, u(y)=M\}} u(y)dy + \frac{1}{\alpha(n)r^n}\int_{\{y:\, u(y)<M\}} M$$
$$= M$$

This shows that u(x)=M a.e. and by continuity everywhere on B(x,r). Fix x_0 and repeat this argument for each $r < dist\{x_0, \partial U\}$. As $r \to dist\{x_0, \partial U\}$, $dist\{B(x_0, r), \partial U\} \to 0$ so for some sequence x_n in U, $inf_n dist\{x_n, \partial U\} = 0$. Since $U \subset \overline{U}$ which is compact, some subsequence x_{n_k} converges to some element x_b in ∂U. Since $u \in C^2(\overline{U})$ the continuity at the boundary yields

$$\lim_{k\to\infty} u(x_{n_k}) = u(\lim_{k\to\infty} x_{n_k}) = u(x_b)$$

Since each $x_{n_k} \in B(x_0, r_{n_k})$ for some $r_{n_k} < dist\{x_0, \partial U\}$, $u(x_{n_k}) = M$. The sequence on the left is therefore constant implying $u(x_b) = M$ and that gives the result desired.

2.3 Convex functions map harmonic to subharmonic

Any convex function $\Phi\colon \mathbb{R} \to \mathbb{R}$ that has a second derivative satisfies $\Phi^{''}(x) \geq 0$. The result of this section is to show that for any convex function Φ with a second derivative the function

$$v(x) = \Phi(u(x))$$

is subharmonic when u is harmonic. To do this, compute:

$$v_{x_i}(x) = \Phi^{'}(u(x))u_{x_i}(x)$$
$$v_{x_ix_i} = \Phi^{''}(u(x))u_{x_i}^2 + u_{x_ix_i}\Phi^{'}(u(x))$$
$$\Rightarrow \triangle v(x) = \sum_{i=1}^{n} v_{x_ix_i} =$$
$$\sum_{i=1}^{n} \Phi^{''}(u(x))u_{x_i}^2 + \Phi^{'}(u(x))\triangle u$$
$$= \sum_{i=1}^{n} \Phi^{''}(u(x))u_{x_i}^2 \geq 0$$

Where on the last line the fact that $\Phi^{''}(x) \geq 0$ was used. Multiplying the equality by -1 on each sides gives that v is subharmonic.

2.4 $Du \cdot Du$ is subharmonic

The purpose of this section is to show that the function $Du \cdot Du$ is subharmonic when u is harmonic. To do this, consider first the fact that $\Phi(x) = x^2$ has a positive second derivative everywhere of 2 therefore it is convex. Next use the fact that any harmonic function is C^∞. Differentiate both sides of Laplace's equation to get:

$$(\triangle u)_{x_i} = \triangle u_{x_i} = 0$$

This is Laplace's equation for the function u_{x_i}. By the theorem proven above, $\Phi(u_{x_i})$ is subharmonic. Therefore

$$\begin{aligned}\triangle(Du \cdot Du) &= \triangle \sum_{j=1}^{n} u_{x_j}^2 = \triangle \sum_{j=1}^{n} \Phi(u_{x_j}) \\ &= \sum_{j=1}^{n} \triangle \Phi(u_{x_j}) \geq \sum_{j=1}^{n} 0 = 0\end{aligned}$$