Algebra Comprehensive Exam Solutions
Fall 2004

Instructions: Attempt any five questions, and please provide careful and complete answers. If you attempt more questions, specify which five should be graded.

1. (a) Prove that a group of order $1225 = 7^2 \cdot 5^2$ is abelian.

   (b) List the groups of order 1225 up to isomorphism.

   Solution: (a) In general, the number of p-Sylow subgroups of a finite group G is $1 \mod p$ and divides $|G|$. Let $|G| = 1225$. The number of 5-Sylow subgroups of G is $1 \mod 5$ and divides 49, hence there is a unique 5-Sylow subgroup P which, therefore, is a normal subgroup. Similarly there is a unique, hence normal, 7-Sylow subgroup Q.

   For primes p, groups of order p^2 are abelian, so P and Q above are abelian. Let $x \in P$ and $y \in Q$. Since P and Q are normal, $xyx^{-1}y^{-1} \in P \cap Q = \{e\}$, so $xy = yx$ and it follows that all elements of $G = PQ$ commute.

   (b) The groups of order 1225 are

   $$\frac{\mathbb{Z}}{\langle 5^2 \cdot 7^2 \rangle}, \qquad \frac{\mathbb{Z}}{\langle 5 \rangle} \times \frac{\mathbb{Z}}{\langle 5 \cdot 7^2 \rangle}, \qquad \frac{\mathbb{Z}}{\langle 7 \rangle} \times \frac{\mathbb{Z}}{\langle 5^2 \cdot 7 \rangle}, \qquad \frac{\mathbb{Z}}{\langle 5 \cdot 7 \rangle} \times \frac{\mathbb{Z}}{\langle 5 \cdot 7 \rangle}.$$

2. Let G be a group with identity element e, with the property that for any two elements $x, y \in G \setminus \{e\}$, there exists an automorphism σ of G with $\sigma(x) = y$.

   (a) Prove that all elements of $G \setminus \{e\}$ have the same order.

   (b) If G is finite, prove that it is abelian.

   Solution: (a) This follows since $|x| = |\sigma(x)|$ for any element $x \in G$ and any automorphism σ.

   (b) Let p be a prime dividing $|G|$. There exists $x \in G$ with $|x| = p$, so all elements of $G \setminus \{e\}$ have order p and therefore G is a p-group (i.e., $|G| = p^n$). A p-group has a nontrivial center $Z(G) \neq \{e\}$. But if $y \in Z(G)$ then $\sigma(y) \in Z(G)$ for any automorphism σ, hence $Z(G) = G$.

3. Let $GL_n(\mathbb{C})$ be the multiplicative group of $n \times n$ matrices of complex numbers. Prove that every element of $GL_n(\mathbb{C})$ of finite order is diagonalizable.

   Solution: If $A^k = I$ for $A \in GL_n(\mathbb{C})$, then the minimal polynomial $p(x)$ of A divides $x^k - 1 \in \mathbb{C}[x]$. But $x^k - 1$ has distinct roots in $\mathbb{C}$, hence so does $p(x)$, and therefore A is diagonalizable.

4. Determine all maximal ideals of the ring

   $$\mathbb{Z}[x]/(120, x^3 + 1).$$

   Solution: Maximal ideals of $R = \mathbb{Z}[x]/(120, x^3 + 1)$ correspond to maximal ideals of $\mathbb{Z}[x]$ containing $(120, x^3 + 1)$. Since $120 = 2^3 \cdot 3 \cdot 5$, every maximal ideal of R must contain either 2 or 3 or 5. Now determine the irreducible factors of $x^3 + 1$ over each of $\mathbb{Z}/(2)$, $\mathbb{Z}/(3)$, $\mathbb{Z}/(5)$. The maximal ideals of R are

   $$(2, x+1)R, \quad (2, x^2 + x + 1)R, \qquad (3, x+1)R, \qquad (5, x+1)R, \quad (5, x^2 - x + 1)R.$$

5. For which integers $n \geq 1$ does the polynomial

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \in \mathbb{Q}[x]$$

have multiple roots?

Solution: The derivative of $f(x)$ is

$$f'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!},$$

so $\gcd(f(x), f'(x)) = \gcd(f(x), x^n) = 1$. Hence $f(x)$ and $f'(x)$ are relatively prime for all $n \geq 1$, and so $f(x)$ always has distinct roots.

6. For an integer $n \geq 3$, consider a regular n-sided polygon inscribed in a circle of radius 1. Let $P_1, \ldots, P_n$ be its vertices, and λ_k be the length of the line joining P_n and P_k for $1 \leq k \leq n-1$. Prove that

$$\lambda_1 \cdots \lambda_{n-1} = n.$$

Solution: There is no loss of generality in taking the unit circle in the complex plane and $P_n = 1$. It follows that $\lambda_k = |1 - e^{2\pi ik/n}|$. The elements $e^{2\pi ik/n}$ for $k = 1, \ldots, n-1$ are the distinct n-th roots of unity other than 1, hence are precisely the roots of the polynomial

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

This means that

$$\prod_{k=1}^{n-1} \left(x - e^{2\pi ik/n}\right) = 1 + x + x^2 + \cdots + x^{n-1}.$$

Evaluating this polynomial at $x = 1$, we get

$$\lambda_1 \cdots \lambda_{n-1} = |\prod_{k=1}^{n-1} \left(1 - e^{2\pi ik/n}\right)| = |1 + 1^1 + 1^2 + \cdots + 1^{n-1}| = n.$$

7. Let A be a real $n \times n$ matrix and let

$$M = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\},$$

where $|\lambda|$ denotes the absolute value of the complex number λ.

(a) If A is symmetric, prove that $\|Ax\| \leq M\|x\|$ for all $x \in \mathbb{R}^n$, where $\| \ \|$ denotes the Euclidean norm on $\mathbb{R}^n$.

(b) Is this true if A is not symmetric? Prove or disprove.

Solution: (a) Since A is a real symmetric matrix it has real eigenvalues $\lambda_1, \ldots, \lambda_n$, and corresponding eigenvectors $v_1, \ldots, v_n$ which form an orthonormal basis for $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$, let $x = \sum a_i v_i$. Then

$$\|Ax\|^2 = \|\sum a_i \lambda_i v_i\|^2 = \sum |a_i \lambda_i|^2 \leq M^2 \sum |a_i|^2 = M^2 \|x\|^2.$$

(b) False, e.g. take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$