Algebra Comprehensive Exam
Fall 2005

Instructions: Attempt any **five** questions, and please provide **careful and complete answers with proofs**. If you attempt more questions, **specify** which five should be graded.

1. Let x and y be elements of a group G such that x has order 3, and y is not the identity and has odd order. If $xyx^{-1} = y^{17}$, determine the order of y.

   Solution: Using $x^n y x^{-n} = y^{17^n}$ with $n = 3$, we get $y = y^{4913}$, so the order of y divides $4912 = 2^4 307$. Since the order is an odd integer greater than 1, it must be 307.

2. Let G be a finite group and p the smallest prime dividing the order of G. Let H be a subgroup of index p. Prove that H is normal in G.

   Solution: Let N denote the normalizer of H. Since $H \subset N$, and $|G:N| \leq |G:H|$, it follows $N = G$ or $N = H$. If $N = G$ OK. Otherwise, the orbit of H under conjugation contains p elements. The action of G on that orbit gives a homomorphism from G to the symmetric group S_p. Let K be the kernel of this homomorphism. K is the intersection of the isotropy groups, and the isotropy of H is H, by assumption. So, $K \subset H$. If $K \neq H$, then from $|G:K| = |G:H||H:K| = p|H:K|$ and the fact that only the first power of p divides $p!$, we conclude that some prime dividing $(p-1)!$ also divides $|H:K|$ contrary to the hypothesis on p.

3. Does there exist a field K such that the multiplicative group $K^* = K \setminus \{0\}$ is isomorphic to the Klein-4-group, $\mathbb{Z}_5 \times \mathbb{Z}_7$?

   Solution: No: even though $\mathbb{Z}_5 \times \mathbb{Z}_7 = \mathbb{Z}_3 5$ is cyclic, then K has 36 elements, impossible since finite fields have elements a power of a prime number.

4. Consider the field $K = \mathbb{Q}(\sqrt{2} + \sqrt{7})$. Find the set $\mu(K)$ of all roots of unity K and describe which abelian group it is isomorphic to.

   Solution: K is a real subfield of $\mathbb{C}$, and its intersection with the unit circle around 0 is the set $\mu(K) = \{-1, 1\}$. This is a cyclic abelian group of order 2.

5. Is every ideal of the ring $\mathbb{Z} \times \mathbb{Z}$ a principal ideal? Prove or disprove.

   Solution: In general, every ideal of a ring $R_1 \times R_2$ is of the form $I_1 \times I_2$ for ideals I_i of R_i. Since every ideal of $\mathbb{Z}$ is principal, it follows that every ideal of $\mathbb{Z} \times \mathbb{Z}$ consists of all multiples of some element $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$, hence is principal.

6. Let R denote the set of all periodic sequences of real numbers, ie:

$$R = \{\alpha : \mathbb{N} \to \mathbb{R}, \alpha_{n+d} = \alpha_n \text{ for some } d\}.$$

   R is a ring with point-wise addition and multiplication:

$$(\alpha + \beta)_n = \alpha_n + \beta_n, \quad (\alpha\beta)_n = a_n \beta_n.$$

(a) Is R an integral domain? Justify your answer.
(b) Let $\mathbf{c}$ denote the constant sequence $(c, c, c, \dots)$ for $c \in \mathbb{R}$. Find all solutions in R of the polynomial equation:

$$x^2 - \mathbf{1} = \mathbf{0}$$

Solution: (a) R is not an integral domain since

$$(1, 0, 1, 0, \dots).(0, 1, 0, 1, \dots) = (0, 0, 0, 0, \dots).$$

(b) The equation has solution all periodic sequences with terms ± 1.

7. Let A be a square matrix with complex entries, and ϵ a positive real number. Prove that $\epsilon I + A^* A$ is nonsingular.

   Solution: If $B = \epsilon I + A^* A$, and v a vector, and $\cdot$ the inner product. We can compute $Bv \cdot v = \epsilon v \cdot v + Av \cdot Av \geq \epsilon v \cdot v = 0$ iff $v = 0$.

8. Consider a group homomorphism $f : \mathbb{Z}^4 \to \mathbb{Z}^2$. List, up to isomorphism, all the possibilities for the kernel of f. Hint: It is a finite list.

   Solution: Every subgroup of $\mathbb{Z}^4$ is isomorphic to $\mathbb{Z}^n$ for $n = 0, 1, 2, 3, 4$. It cannot be $0, 1$ since then the image is too big. So, $\mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}^4$ are the three possibilities.