SOME PROBLEMS IN FUNCTIONAL ANALYSIS.

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PROBLEM 1. Prove that a necessary and sufficient condition that the metric space (X, d) be complete is that every nested sequence of nonempty closed sets $(F_i)_{i=1}^{\infty}$ with diameters tending to 0 , has a nonempty intersection:

$$\bigcap_{i=1}^{\infty} F_i \neq \emptyset.$$

PROBLEM 2. Let f be acontinuous function such that $f : X \to Y$ where X and Y are metric spaces. Suppose A is a compact subset of X, prove that $f(A)$ is a compact subset of Y.

PROBLEM 3. We know that if (X, d) is a metric space. Then $A \subset X$ is compact only if A is closed and bounded. In R^n we know that the converse is also true which is precisely the statement of the Heine-Borel Theorem. Give a counterexample to show that the converse is not necessarily true if we have an arbitrary metric space.

PROBLEM 4. For each n let $f_n : R \to R$ be a differentiable function. Suppose also that for each n and x we have $|f_n^{'|\leq 1}$. Show that if for all x $\lim_{n\to\infty} f_n(x) = g(x)$ then $g : R \to R$ is a continuous function.

PROBLEM 5. Let $f_n : [0, 1] \to [0, \infty)$ be continuous for $n = 1, 2, ...$ Suppose

$$(*) f_1(x) \geq f_2(x) \geq f_3 \geq ...$$

for $x \in [0, 1]$ and let $f(x) = \lim_{n\to\infty} f_n(x), M = \sup_{x\in[0,1]} f(x)$.
a.)Show that there exists $t \in [0, 1]$ such that $f(t) = M$.
b.)Does the conclusion remain valid if we replace the condition $(*)$ by the follwing condition? Suppose there exists n_x such that $f_n(x) \geq f_{n+1}(x)$ for all $x \in [0, 1]$ and $n > n_x$.

PROBLEM 6. Let (f_n) be a nondecreasing sequence of functions from $[0, 1]$ to $[0, 1]$. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ and suppose also that f is continuous. Prove that $f_n \to f$ uniformly.

PROBLEM 7. Suppose $f : R^n \to R^m$ satisfies following properties.
(i) For all compact subsets K of R^n, $f(K)$ is also compact.

(ii) For decreasing sequence (K_n) of compact subsets of R^n one has $f(\cap_n K_n) = \cap_n f(K_n)$. Show that f is a continuous function on R^n.

PROBLEM 8. Suppose X, Y are metric spaces and $f : X \to Y$ is a continuous function. For all n let K_n be nonempty compact sets such that $K_{n+1} \subset K_n$ and let $K = \cap K_n$. Show that $f(K) = \cap f(K_n)$.

PROBLEM 9. Show that every orthonormal sequence in an infinite-dimensional Hilbert space converges weakly to 0.

PROBLEM 10. Let X be a topological space, Y a Hausdorff space, and f, g continuous functions from X to Y.
a.) Show that $\{x : f(x) = g(x)\}$ is closed.
b.) Show that if $f = g$ on a dense subset of X, then $f = g$ on all of X.

PROBLEM 11. Suppose (f_n) is a sequence of continuous functions such that $f_n : [0, 1] \to R$ and as $n, m \to \infty$,

$$\int_0^1 (f_n(x) - f_m(x))^2 dx \to 0.$$

Suppose also that $K : [0, 1] \times [0, 1] \to R$ is continuous. Define $g_n : [0, 1] \to R$ by

$$g_n = \int_0^1 K(x, y) f_n(y) dy.$$

Show that (g_n) is uniformly convergent sequence.

PROBLEM 12. Let C denote the space of all bounded continuous functions on the real line R equipped with the supremum norm. Let S be the subspace of C consisting of functions f such that

$$\lim_{n\to\infty} f(x)$$

exists.
a.) Is S a closed linear subspace of C?
b.)Show that there is a bounded linear functional L on C so that

$$L(f) = \lim_{x\to\infty} f(x)$$

for all $f \in S$.
c.) Is there a bounded Borel measure μ so that $L(f) = \int_R f d\mu$ for all $f \in C$?

PROBLEM 13. Let X be a normed linear space. Show that if S is an open subspace of X, then $S = X$.

PROBLEM 14. Suppose that A and K are closed subsets of an additive topological group G, prove that if K is compact then $A + K$ is closed.

PROBLEM 15. Let X and Y be normed vector spaces, and let $L : X \to Y$ be linear and bounded.

a.) Show that $N(L) = \{x \in X : L(x) = 0\}$ is a closed subspace of X.
b.) Now let $X = C$, the set of complex numbers. Shiw that $R(L) = \{L(x) : x \in C\}$ is a closed subspace of Y. Hint: Every vector in C is a scalr multiple of 1.

PROBLEM 16. Define $L : l^2 \to l^2$ by $L(x_1, x_2, ...) = (x_2, x_3, ...)$. Prove that L is bounded and find $||L||$. Is L injective?

PROBLEM 17. Let X be a normed space, and suppose that $x_n \to x \in X$. Show that there exists a subsequence (x_{nk}) such that

$$\sum_{k=1}^{\infty} ||x - x_{nk}|| < \infty.$$