GROUP THEORY Summer 2003

SOLUTIONS TO SOME PROBLEMS

Warning:These solutions may contain errors!!

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PROBLEM 1. Suppose that G is a group of even order. Prove that G contains at least one element $a \neq e$ such that $a.a = e$.

SOLUTION. Let $|G|$ denote the number of elements in the group G. Suppose there exists no $a \in G$ such that $a.a = e \Rightarrow a \neq a^{-1}$. Then, we have

$$G - \{e\} = \bigcup_{a \in (G-\{e\})} \{a, a^{-1}\}.$$

Observe that $\{a, a^{-1}\} \cap \{b, b^{-1}\} \neq \emptyset \Rightarrow (a = b \text{ or } a = b^{-1})$ or $(a^{-1} = b \text{ or } a^{-1} = b^{-1})$. Thus,in this case we have, $\{a, a^{-1}\} = \{b, b^{-1}\}$ and by assumption $|\{a, a^{-1}\}| = 2$. Then, we have the following disjoint union

$$G - \{e\} = \{a_1, a_1^{-1}\} \cup ... \cup \{a_k, a_k^{-1}\}.$$

This implies that $|G - \{e\}|$ is an even number so that $|G|$ is an odd number, which is a contradiction.

PROBLEM 2. Let G be a group such that every element of G is its own inverse. i.e. $a.a = e$ for all $a \in G$. Prove that G is abelian.

SOLUTION. We need to show that $a.b = b.a$ for all $a, b \in G$. By assumption $a^2 = b^2 = e = (a.b)^2$. We have $ab = a^{-1}b^{-1}$ as $a = a^{-1}$ and $b = b^{-1}$. This gives,

$$a.b = a^{-1}.b^{-1} = (b.a)^{-1} = b.a.$$

Since a, b are arbitrary, we conclude that G is abelian.

PROBLEM 3. Let p, q be distinct primes such that $p < q$ and $q \neq 1 (modp)$. Let G be a any group of order pq. Show that G must be a cyclic group.

SOLUTION. Let n_p be the number of Sylow p subgroups of G and let n_q be the number of sylow q subgroups of G. Then we have $n_p|q \Rightarrow n_p \in \{1, q\}$. Since $q \neq 1(mod p)$ we conclude that $n_p = 1$. Similarly, $n_q|p \Rightarrow n_q \in \{1, p\}$. If $n_q = p$ then $p \equiv 1(mod q) \Rightarrow q < p$. But this is not possible so that $n_q = 1$.

Let P be the unique sylow p subgroup of G and let Q be the unique sylow q subgroup of G. Now, $P \cap Q = \{e\}$. Since $|P \cap Q|||P| = p$ and $|P \cap Q|||Q| = q \Rightarrow |P \cap Q| = 1$. Thus, we have $P \trianglelefteq G, Q \trianglelefteq G$ and $G = PQ \Rightarrow G = P \times Q \cong Z_p \times Z_q$.