REAL ANALYSIS Spring 2003

SOLUTIONS TO SOME PROBLEMS

Warning:These solutions may contain errors!!

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PROBLEM 1. Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, ...$; $f_1 \geq f_2 \geq f_3 \geq ... \geq 0$; $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.
a)Give a counterexample to show that we do not have generally the following result. $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.
b) Without changing any other assumptions just add one more assumption and prove that the conclusion is satisfied in this case.

SOLUTION.
a)The standard example is the following.Let $f_n = 1_{[n,\infty)}$ for $n = 1, 2, 3, ...$,where 1_X represents the characteristic function of the set X.Then one can easily show that $f = 0$ but $\lim_{n\to\infty} \int_X f_n d\mu$ does not exist.
Consider $f_n = \frac{1}{n} 1_{[n,\infty)}$ for $n = 1, 2, 3, ...$,where 1_X represents the characteristic function of the set X. Then one can easily checks that $f = 0$ but $\lim_{n\to\infty} \int_X f_n d\mu$ does not exist.
Here is another example:In this example we give a counterexample to the case where we have strict inequality.
Let $f = 0$.Let $f_n = \frac{1}{n}$ if $-\infty < x \leq n$ and $f_n = \frac{1}{2} + \frac{1}{n}$ if $n < x < \infty$.Thus f_n strictly decreases to $f = 0$ but $\int_X f_n d\mu = \infty$ for all n yet $\int_X f_n d\mu = 0$.
b) If we assume that $f_1 \in L_1(X, \mu)$ then the conclusion is satisfied.Here is the $proof$:
Consider the sequence $g_n = f_1 - f_n$ since $f_1 > f_2 > f_3 > ... > 0$ we have $g_{n+1} > g_n > 0$ for all $n = 1, 2, 3, ...$.Also $g_n(x) \to f_1(x) - f(x)$ as $n \to \infty$.Now we apply the Monotone Convergence Theorem to the sequence g_n and get $\lim_{n\to\infty} \int_X g_n d\mu = \int_X \lim_{n\to\infty} g_n d\mu$.
But this means that $\int_X f_1 d\mu - \lim_{n\to\infty} \int f_n d\mu = \int f_1 d\mu - \int f d\mu$.Now since $f_1 \in L_1(X, \mu)$ we can delete the term $\int f_1 d\mu$ from both sides and get $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

PROBLEM 2. Suppose $\mu(X) < \infty$, f_n is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$. Show by a counterexample that the conclusion is not valid if we omit $\mu(X) < \infty$.

SOLUTION. We first establish that the function f is integrable.For this note that we can choose N so large that $|f_N - f| < \epsilon/2$ and for any given $\epsilon > 0$.Then we have the following estimate :
$\int_X |f|d\mu \leq \int_X |f_N - f|d\mu + \int_X |f_N|d\mu \leq (\epsilon/2)\mu(X) + M\mu(X)$ (*) where $M < \infty$ is a bound for f_N(by assumption f_n's are bounded for all n).Since $\mu(X) < \infty$ the right hand side of (*) is finite,which shows that f is integrable.Now we have N so that $|f_N - f| < \epsilon/2$ for fixed ϵ.Now consider the following estimate for large enough n:
$\int_X |f_n - f|d\mu \leq \int_X |f - f_N|d\mu + \int_X |f_n - f_N|d\mu < (\epsilon/2)\mu(X) + (\epsilon/2)\mu(X)$.
Since $\epsilon > 0$ is arbitrary and $\mu(X) < \infty$ we have that $lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.
For a counterexample to the case $\mu(X) = \infty$,consider $f_n = 1/n1_[0, n)$ where 1_X represents the charactristic function of the set X. Then one can easily checks that

$$\int_X f_n d\mu = 1, \forall n$$

but $f = 0$.Hence the conclusion does not necessarily hold when $\mu(X) = \infty$.

PROBLEM 3. Suppose $f_1 \in L_1(X, \mu)$.Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f|d\mu < \epsilon$ whenever $\mu(E) < \delta$.

SOLUTION. Define $f_n = |f| \wedge n$. i.e. $f_n = |f|$, if $|f| \leq n$ and $f_n = n$,if $|f| > n$.Then $f_n \to |f|$ as $n \to \infty$. Therefore we can use Monotone Convergence Theorem as $f_{n+1} \geq f_n$ and $f_n \geq 0$.Now choose N so that $\int_E |f - f_N| < \epsilon/2$ for given $\epsilon > 0$.Then we have the following estimate.
$\int_E |f|d\mu < \int_E |f - f_N|d\mu + \int_E |f_N|d\mu < (\epsilon/2) + N\mu(E) < \epsilon/2 + N\mu(E) < \epsilon$ whenever $\mu(E) < \delta < \epsilon/(2N)$.Thus the assertion is proved.

PROBLEM 4. Let X be an uncountable set,let M be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second case.Prove that M is a σ-algebra in X and that μ is a measure on M.

SOLUTION. The solution is obtained by direct applications of definitions.
M is a σ-algebra in X: Clearly $\emptyset$ and X are in M.Also suppose F is a member of M then we have two cases.Either F is countable , in this case $(F^c)^c = F$ is countable which shows that F^c is in M or F^c is countable which shows that F^c is in M. The most interesting part is to show that M is closed under countable unions.Suppose $E_1, E_2, E_3, ...$ is contable collection of sets each of which is in M.Then we have to consider the following cases:
1.) Suppose each E_i is countable.In this case the union will be countable and hence it will lie in M.
2.) Suppose now that there exists E_k for some k such that E is uncountable.Then E^c is countable.And by the De Morgan's Law we have $(\cup E_i)^c \subset E_k^c$ and this shows that the complement of the union is countable and so the union lies in M.
Therefore M is a σ-algebra in X.
μ is a measure on M :We need to show the following assertions.

1.)$\mu(\emptyset) = 0$
2.)Countable additivity.*i.e.*If $E_1, E_2, E_3, ...$ is contable collection of sets each of which is in M then $\mu(\cup E_i) = \Sigma\mu(E_i)$.
The first assertion is obvious as $\emptyset$ is countable.So by definition $\mu(\emptyset) = 0$.
The interesting part is the second assertion.Suppose first that each of E_i is countable. In this case the union of these sets is also countable.So we have $\mu(\cup E_i) = 0$ by definition.On the other hand since each E_i is countable $\mu(E_i) = 0$ for all i.Thus the second assertion above holds in this case.
Now suppose that $\exists$ k such that E_k is uncountable.Then E_k^c is countable and by the De Morgan's Law used above we again have $(\cup E_i)^c \subset E_k^c$ which shows that $\mu(\cup E_i) = 1$.If we consider the summation $\Sigma\mu(E_i)$ we see that it is equal to 1 since the only term that is nonzero(1) is $\mu(E_k)$.So we again have the validity of the second assertion.
Thus μ defined above is a measure in M.

PROBLEM 5. Let E_k be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.(*)$$

a.)Then show that almost all $x \in X$ lie in at most finitely many sets E_k.
b.)Is the conclusion still valid if we omit the condition $(*)$?

SOLUTION. This is known as the Borel-Contelli's Lemma.There are two ways to do the part a.)
First Proof:If A is the set of all x which lie in infinitely many E_k, we need to prove that $\mu(A) = 0$.Put

$$g(x) = \sum_{k=1}^{\infty} 1_{E_k}(x), (x \in X)$$

where 1_K represents the characteristic function of the set K.Observe that for each x, each term in this series is either 1 or 0.Hence $x \in X$ if and only if $g(x) = \infty$.But we know that the integral of g is equal to the sum in $(*)$.Thus $g \in L^1(\mu)$,and so $g(x) < \infty$ a.e.
Second Proof:From set theory we see that the set we are looking for is $A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$. Define $F_n = \cup_{k=n}^{\infty} E_k$.Then clearly $F_{n+1} \subset F_n$.Thus

$$\mu(A) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} E_k) \leq lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k)(**)$$

But the last term in $(**)$ is the limit of the remainder term of the series $(*)$ which is finite.Hence it goes to 0.Thus the assertion is proved.
b.)As you guess the conclusion is not valid if we omit the finiteness condition in $(*)$.Here is a simple example.Take $E_k = (-\infty, 1/n)$.Then the sum in $(*)$ is equal to ∞.And $A = (-\infty, 0]$ which has measure ∞.

PROBLEM 6. Find a sequence (f_n) of Borel measurable functions on R which decreses uniformly to 0 on R , but $\int f_n dm = \infty$ for all n.Also,find a sequence (g_n) of Borel measurable functions on $[0,1]$ such that $g_n \to 0$ pointwise but $\int g_n dm = 1$ for all n. (here m is the Lebesgue measure!)

SOLUTION. For the first part $f_n = 1/n 1_{[1,\infty)}$ works(easy Calculus exercise!!).For the second part one sees that $g_n = n1_{[0,1/n]}$ satisfies all the assertions(again this is easy to verify).

PROBLEM 7. Show that Monotone Convergence Theorem can be proved as a corollary of the Fatou's lemma.

SOLUTION. Applly the Fatou's lemma to the following sequences $(f+f_n)$ and $(f-f_n)$. Since $f_n \uparrow f$ both of these sequences are non-negative. Hence, application of the Fatou's lemma to the sequence $(f+f_n)$ gives $\liminf \int f_n \geq \int f$. And application of Fatou's lemma to the sequence $(f-f_n)$ gives $\limsup \int f_n \leq \int f$.Combination of these two inequalities proves the Monotone Convergence Theorem.

PROBLEM 8. Let $f \in L^+$ and $\int f < \infty$, then show that the set $\{x : f(x) > 0\}$ is $\sigma-$finite.

SOLUTION. This follows from the following equality.

$$\cup_{n=1}^{\infty}\{x : f(x) > 1/n\} = \{x : f(x) > 0\}.$$

Each of the sets on the left hand side of this equality is of finite measure , since otherwise f would not have finite integreal.And this shows that the set in question is the union of sets of finite measure.

PROBLEM 9.
a.)If f is nonnegative and integrable on A, then show that

$$\mu(\{x : x \in A, f(x) \geq c\}) \leq 1/c \int_A f(x)d\mu$$

b.)If $\int_A |f(x)|d\mu = 0$, prove that $f(x) = 0$ a.e.

SOLUTION.
a.)This is known as the Chebyshev's inequality.If $A_1 = \{x : x \in A, f(x) \geq c\}$, then

$$\int_A f(x)d\mu = \int_{A_1} f(x)d\mu + \int_{A-A_1} f(x)d\mu \geq \int_{A_1} f(x)d\mu \geq c\mu(A_1).$$

This proves the result.
b.)By the Chebyshev's inequality,

$$\mu(\{x : x \in A, f(x) \geq 1/n\}) \leq n \int_A f(x)d\mu = 0, \forall n = 1, 2, ...$$

Therefore,

$$\mu(\{x : x \in A, f(x) \neq 0\}) \leq \sum_{n=1}^{\infty} \mu(\{x : x \in A, |f(x)| \geq 1/n\}) = 0.$$

And this clearly proves the desired result.

PROBLEM 10.
a.)Consider a measure space (X, μ) with a finite,positive,finitely additive measure μ. Prove that μ is countably additive if and only if it satisfies the following condition. If A_n is a decresing sequence of sets with empty intersection then

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

b.)Now suppose that X is locally compact Hausdorff space, that Br is the Borel $\sigma-$algebra, and that μ is finite, positive, finitely additive measure on Br.Suppose moreover that μ is regular, that is for each $B \in Br$ we have,

$$\mu(B) = \sup_K \{\mu(K) : K \subseteq B, K - \mathit{compact}\}$$

Prove that μ is countably additive.

SOLUTION.
a.)Sufficency:Let (B_n) be countably many measurable sets which are mutually disjoint.Let $A_n = \cup_{i=n+1}^{\infty} B_i$.Then $\cap_{n=1}^{\infty} A_n = \emptyset$.We have

$$\mu(\cup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \{\mu(\cup_{i=1}^{n} B_i) + \mu(\cup_{i=n+1}^{\infty} B_i)\}$$

$$= \lim_{n\to\infty} \{\sum_{i=1}^{\infty} \mu(B_i) + \mu(A_n)\} = \sum_{i=1}^{\infty} \mu(B_i).$$

Therefore μ is a measure.The necessity is obvious.
b.) If μ is not countably additive ,by $a.$) there is a decreasing sequence (A_n) of measurable sets with empty intersection such that

$$\lim_{n\to\infty} \mu(A_n) = \inf \mu(A_n) > 0.$$

For each n there exists K_n contained in A_n, such that

$$\mu(A_n) < \mu(K_n) + 1/2^{n+1} \inf_i \mu(A_i).$$

Then

$$\mu(A_n - \cap_{i=1}^{n} K_i) \leq \sum_{i=1}^{n} \mu(A_i - K_i) < 1/2 \inf_i \mu(A_i)$$

which implies that $\mu(\cap_{i=1}^n K_i) \neq 0$ and therefore $\cap_{i=1}^n K_i \neq \emptyset$.
Thus $\{\cap_{i=1}^n K_i : n \in N\}$ is a decreasing sequence of nonempty compact subsets in the compact space K_1.So $\cap_{i=1}^\infty K_i \neq \emptyset$,which contradicts the fact that $\cap_{n=1}^\infty A_n = \emptyset$.

PROBLEM 11. Let λ be Lebesgue measure on R. Show that for any Lebesgue measurable set $E \subset R$ with $\lambda(E) = 1$, there is a Lebesgue measurable set $A \subset E$ with $\lambda(A) = 1/2$.

SOLUTION. Define the function $f : R \to [0,1]$ by $f(x) = \lambda(E \cap (-\infty, x])$, where $x \in R$.It is continuous by the following inequality

$$|f(x) - f(y)| \leq |x - y|,$$

where$x, y \in R$.Since $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$, there is a point $x_0 \in R$ such that $f(x_0) = 1/2$.Put $A = E \cap (-\infty, x_0]$.

PROBLEM 12. Let m be a countably additive measure defined for all sets in a $\sigma-$algebra M.
a.)If A and B are two sets in M with $A \subset B$, then show that $m(A) \leq m(B)$.
b.)Let (B_n) be any sequence of sets in M.Then show that $m(\cup_{n=1}^\infty B_n) \leq \sum_{n=1}^\infty m(B_n)$.

SOLUTION. These are almost trivial statements.
a.)We have $B = A \cup (B - A)$ and using countable additivity of m by taking other sets to be empty gives, $m(B) = m(A) + m(B - A)$.But $m(B - A) \geq 0$. So the result follows.
b.)By set theory we can find a mutually disjoint sequence (A_n) such that $\cup_{n=1}^\infty A_n = \cup_{n=1}^\infty B_n$.So we have,

$$m(\cup_{n=1}^\infty B_n) = m(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty m(A_n) \leq \sum_{n=1}^\infty m(B_n),$$

where the second equality follows from the countable additivity of m and the last inequlity follows from the fact that each term in the sum on the left is less than or equal to the corresponding term on the right.i.e.$A_n \subseteq B_n, \forall n$ hence by part $a.), m(A_n) \leq m(B_n), \forall n$. So the result follows.

PROBLEM 13. a.)Let (E_n) be an infinite decreasing sequence of Lebesgue measurable sets, that is, a sequence with $E_{n+1} \subset E_n$for each n. Let $m(E_1)$ be finite, where m is the Lebesgue meausre.Then show that $m(\cap_{i=1}^\infty E_i) = \lim_{n\to\infty} m(E_n)$.
b.)Show by acounterexample that we can not omit the condition $m(E_1)$ is finite.

SOLUTION.
a.) Let $E = \cap_{i=1}^\infty E_i$, and let $F_i = E_i - E_{i+1}$.Then $E_1 - E = \cup_{i=1}^\infty F_i$, and the sets F_i are pairwise disjoint.Hence,

$$m(E_1 - E) = \sum_{i=1}^\infty m(F_i) = \sum_{i=1}^\infty m(E_i - E_{i+1}).$$

But $m(E_1) = m(E_1) + m(E_1 - E)$, and $m(E_i) = m(E_{i+1}) + m(E_i - E_{i+1})$,since $E \subset E_1$ and $E_{i+1} \subset E_i$. Since $m(E_i) \leq m(E_1) < \infty$, we have $m(E_1 - E) = m(E_1) - m(E)$ and $m(E_i - E_{i+1}) = m(E_i) - m(E_{i+1})$.Thus

$$m(E_1) - m(E) = \sum_{i=1}^{\infty}(m(E_i) - m(E_{i+1})$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(m(E_i) - m(E_{i+1})$$

$$= \lim_{n\to\infty}(m(E_1) - m(E_n)$$

$$= m(E_1) - \lim_{n\to\infty} m(E_n$$

Since $m(E_1) < \infty$, we have $m(E) = \lim_{n\to\infty} m(E_n)$

b.)Let $E_n = (n, \infty)$.Then $m(E_n) = \infty$ for all n,whereas $\cap_{n=1}^{\infty} E_n = \emptyset$.

PROBLEM 14.

a.)Show that we may have strict inequality in Fatou's Lemma.

b.)Show that Monotone Convergence Theorem need not hold for decreasing sequences of functions.

SOLUTION.

a.)Consider the sequence (f_n) defined by $f_n(x) = 1$ if $n \leq x < n+1$, with $f_n(x) = 0$ otherwise.Then, $\liminf f(x) = 0$ but $\int f_n(x) = 1, \forall n$. Hence the strict inequality holds.

b.)Let $f_n(x) = 0$ if $x < n$ and $f_n(x) = 1$ if $x \geq n$.i.e.$f_n(x) = 1_[n, \infty)$ where 1_A represents the characteristic function of the set A. Now clearly $f_n \searrow 0$ but $\lim_{n\to\infty}\int f_n$ is undefined.Hence M.C.T. does not hold in this case.

PROBLEM 15. Let (f_n) be a sequence of nonnegative measurable functions that converge to f, and suppose that $f_n \leq f, \forall n$. Then show that $\int f = \lim \int f_n$.

SOLUTION. We know that $f_n \to f$ and $f_n \leq f, \forall n$. Therefore we can choose a subsequence f_{nk} such that $f_{nk} \nearrow f$ as $k \to \infty$. Thus an application of the Monotone Convergence Theorem to the sequence f_{nk} gives the result.

PROBLEM 16. Suppose $A \subset R$ is Lebesgue measurable and assume that

$$m(A \cap (a, b)) \leq (b - a)/2$$

for any $a, b \in R, a < b$.Prove that $m(A) = 0$.

SOLUTION. If $m(A) \neq 0$ there is an n such that $m(A \cap (n, n+1) \neq 0$.There is an open set U in $(n, n+1)$ such that

$$A \cap (n, n+1) \subseteq U \subseteq (n, n+1)$$

and $m(U) < m(A \cap (n, n+1)) + \epsilon$,where $\epsilon < m(A \cap (n, n+1))$.
There are at most countably many disjoint intervals $(a_j, b_j)'s$ such that $U = \cup_j (a_j, b_j)$. Then $A \cap (n, n+1) = \cup A \cap (a_j, b_j)$.We have

$$m(A \cap (n, n+1)) = \sum_j m(A \cap (a_j, b_j) \leq \sum_j (b_j - a_j)/2 = 1/2 m(U) < 1/2(m(A \cap (n, n+1)) + \epsilon)$$

which gives $m(A \cap (n, n+1)) < \epsilon$, a contradiction.

PROBLEM 17. Choose $0 < \lambda < 1$ and construct the Cantor set K_λ as follows:Remove from $[0, 1]$ its middle part of length λ; we are left with two intervals L_1 and L_2. Remove from each of them their middle parts of length $\lambda |I_i|, i = 1, 2$, etc and keep doing this ad infimum. We are left with the set K_λ. Prove that the set K_λ has Lebesgue measure 0.

SOLUTION.$Calim$: For any $n \in N$, the total length of intervals removed in the n'th step is $\lambda(1-\lambda)^{n-1}$.
The claim holds for $n = 1$. Assume that it holds for $k \leq n$. Then the total length of intervals removed in the $k + 1'th$ step is

$$\lambda(1 - \sum_{i=1}^{k} \lambda(1-\lambda)^{i-1}) = \lambda(1-\lambda)^k.$$

By induction the claim holds for any $n \in N$.
It follows that the Lebesgue measure of K_λ is

$$1 - \sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} = 0.$$

PROBLEM 18. Let $A \subset [0, 1]$ measurable set of positive measure.Show that there exist two points $x' \neq x''$ in A with $x' - x''$ rational.

SOLUTION. Denote all rational numbers in $[-1, 1]$ by $r_1, r_2, ..., r_n, ...$Denote $A_n = \{x + r_n : x \in A\}$.Then $m(A_n) = m(A) > 0$. $A_n \subset [-1, 2]$.Thus,

$$\cup_{i=1}^{\infty} A_n \subset [-1, 2].$$

Suppose that $A_n \cap A_m = \emptyset$ if $n \neq m$. Then

$$\sum_{n=1}^{\infty} m(A_n) \leq m([-1, 2]) = 3,$$

which contradics $m(A) > 0$. Therefore there must be some n, m such that $A_n \cap A_m \neq \emptyset$. Take $z \in A_n \cap A_m$. Then we can find $x', x'' \in A$ such that

$$z = x' + r_n = x'' + r_m.$$

Thus $x' - x'' = r_m - r_n$.

PROBLEM 19. Let $f : R^n \to R$ be an arbitrary function having the property that for each $\epsilon > 0$, there is an open set U with $\lambda(U) < \epsilon$ such that f is continuous on $R^n - U$(in the relative topology).Prove that f is measurable.

SOLUTION. Let U_k be an open set such that $\lambda(U_k) < 1/k$ and f is continuous on $R^n - U_k$ .Let $f_k = f1_{R^n - U_k}$ (where 1_A represents the characteristic function of the set A), thenf_k is measurable.For any $\epsilon > 0$,

$$m^*(\{x : |f_k - f|(x) \geq \epsilon\}) = m^*(\{x \in U_k : |f(x)| \geq \epsilon\}) \leq 1/k.$$

It follows that (f_k) converges to f in measure.Since Lebesgue measure is complete f is measurable.

PROBLEM 20. Prove or disprove that composition of two Lebesgue integrable functions with compact support $f, g : R \to R$ is still integable.

SOLUTION. It is not true.For example, let$f(x) = 1_{\{0\}}(x)$ and $g(x) = 1_{\{0,1\}}(x)$, where 1_A represents the characteristic function of the set A.Then f and g are integrable functions with compact support. But,since $g \circ f \equiv 1$, the function $g \circ f$ is not integrable.

PROBLEM 21. Let (X, M, μ) be a positive measure space with $\mu(X) < \infty$. Show that a measurable function $f : X \to [0, \infty)$ is integrable (i.e. one has $\int_X f d\mu < \infty$) if and only if the series

$$\sum_{n=0}^{\infty} \mu(\{x : f(x) \geq n\})$$

converges.

SOLUTION. Suppose f is integrable. Then

$$\sum_{n=0}^{\infty} \mu(\{x : f(x) \geq n\}) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \mu(\{x : m \leq f(x) < m+1\})$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{m} \mu(\{x : m \leq f(x) < m+1\}) = \sum_{m=0}^{\infty} (m+1)\mu(\{x : m \leq f(x) < m+1\})$$

$$\sum_{m=0}^{\infty} m\mu(\{x : m \leq f(x) < m+1\}) + \sum_{m=0}^{\infty} \mu(\{x : m \leq f(x) < m+1\})$$

$$\sum_{m=0}^{\infty} \int_{\{x: m \leq f(x) < m+1\}} f(x) d\mu(x) + \mu(X) = \int_X (f+1) d\mu < \infty$$

Conversely,

$$\int_X f d\mu = \sum_{m=0}^{\infty} \int_{\{x: m \leq f(x) < m+1\}} f(x) d\mu(x)$$

$$\leq \sum_{m=0}^{\infty}(m+1)\mu(\{x : m \leq f(x) < m+1\}) = \sum_{n=0}^{\infty}\mu(\{x : f(x) \geq n\}) < \infty$$

which shows that f is integrable.

PROBLEM 22.
a.) Is there a Borel measure μ (positive or complex) on R with the property that

$$\int_R f d\mu = f(0)$$

for all continuous $f : R \to C$ of compact support? Justify.
b.) Is there a Borel measure μ (positive or complex) on R with the property that

$$\int_R f d\mu = f'(0)$$

for all continuous $f : R \to C$ of compact support? Justify.

SOLUTION.
a.) Yes. Let $\mu(E) = 1_E(0)$, where 1_A represents the characteristic function of the set A, for any Borel set E.
b.) No. If there were such a Borel measure , let $\Phi \geq 0$ be a continuously differentiable function of compact support, taking value 1 on $[-1,1]$.Then a contradiction occurs from the following limits.

$$\lim_{n\to\infty}\int_R \Phi(t)e^{t/n}dt = \int_R \Phi(t)dt > 0$$

and

$$\lim_{n\to\infty}(\Phi(t)e^{t/n})'|_{t=0} = \lim_{n\to\infty}(e^{t/n}/n)|_{t=0} = 0.$$

PROBLEM 23. Let f_n be a sequence of real-valued functions in $L^1(R)$ and suppose that for some $f \in L^1(R)$

$$\int_{-\infty}^{\infty}|f_n(t) - f(t)|dt \leq 1/n^2, n \geq 1.$$

Prove that $f_n \to f$ almost everywhere with respect to Lebesgue measure.

SOLUTION. Since

$$\sup_n \int \sum_{k=1}^{n}|f_{k+1} - f_k|dt \leq \sum_{k=1}^{\infty}(1/(k+1)^2 + 1/k^2) < \infty$$

by Levi's theorem there is a measurable set E of measure 0 such that for any $t \in R - E$,

$$\sup_n \sum_{k=1}^{N}|f_{k+1} - f_k|(t) < \infty.$$

Therefore for any $t \in R - E$,

$$f_n(t) = f_1(t) + \sum_{k=2}^{n} (f_k - f_{k-1})(t)$$

converges. It follows that $f_n \to f$ almost everywhere.

PROBLEM 24. Let m denote the Lebesgue measure on $[0,1]$ and let (f_n) be a sequence in $L^1(m)$ and h a non-negative element of $L^1(m)$.Suppose that
i.) $\int f_n g dm \to 0$ for each $g \in C([0,1])$ and
ii.) $|f_n| \leq h$ for all n.
Show that $\int_A f_n dm \to 0$ for each Borel subset $A \subset [0,1]$.

SOLUTION. For any $\epsilon > 0$, there is a $\delta > 0$ such that $\int_E h dm < \epsilon$, whenever $m(E) < \delta$. For such a δ there are a compact set K and an open set U such that $(1) K \subseteq A \subseteq U$ and $(2) m(U - K) < \delta$. There is a continuous function $g : [0,1] \to R$ such that $(3) 0 \leq g \leq 1, (4) g = 1$ on K and $(5) g = 0$ outside U. Then we have

$$\limsup_{n\to\infty} \int_A f_n dm = \limsup_{n\to\infty} |\int_0^1 f_n 1_A dm|$$

$$\leq \limsup_{n\to\infty} |\int_A f_n g dm| + |\int_0^1 f_n (1_A - g) dm|$$

$$\leq \limsup_{n\to\infty} |\int_A f_n g dm| + |\int_0^1 h 1_{U-K} dm| \leq \epsilon$$

It follows that $\lim_{n\to\infty} \int_A f_n dm = 0$.
$NOTE$: Here 1_B represents the characteristic function of the set B.

PROBLEM 25.
a.)Prove the Lebesgue Dominated Convergence Theorem.
b.)Here is a version of Lebesgue Dominated Convergence Theorem which is some kind of extension of it.Prove this.
Let (g_n) be asequence of integrable functions which converges a.e. to an integarble function g.Let (f_n) be asequence of measurable functions such that $|f_n| \leq g_n$ and (f_n) converges to f a.e. If $\int g = \lim \int g_n$, then $\int f = \lim \int f_n$.
c.) Show that under hypotheses of the part b.) we have $\int |f_n - f| \to 0$ as $n \to \infty$.
d.)Let (f_n) be asequence of integrable functions such that $f_n \to f$ a.e. with f is integrable.Then show that
$\int |f - f_n| \to 0$ as $n \to \infty$ if and only if $\int |f_n| \to \int |f|$ as $n \to \infty$.

SOLUTION.
a.) First let us state the theorem properly.
Let g be integrable over E and let (f_n) be a sequence of measurable functions such that

$|f_n| \le g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Then

$$\int_E f = \lim \int_E f_n.$$

Proof :The function $g - f_n$ is nonnegative so by Fatou's Lemma we have

$$\int_E (g - f_n) \le \liminf \int_E (g - f_n).$$

Since $|f| \le g, f$ is integrable, and we have

$$\int_E g - \int_E f \le \int_E g - \limsup \int_E f_n,$$

whence

$$\int_E f \ge \limsup \int_E f_n.$$

Similarly, considering $g + f_n$, we get

$$\int_E f \le \liminf \int_E f_n,$$

and this completes the proof.
b.) We will try to use the same idea as in the above proof. Take $h_n := g_n - f_n$, by noting that $h_n \ge 0$ and $k_n := g_n + f_n$.Applying the Foatou's lemma to these sequences we get the follwing inequalities combination of which proves the result.
Application of Fatou's lemma to h_n gives $\limsup \int f_n \le \int f$ and application of the Fatou's lemma to k_n gives $\liminf \int f_n \ge \int f$.
c.)Take $f = 0, g_n = |f_n| + |f|$ and $g = 2|f|$ in part b.).Note that now our sequence is $|f_n - f|$ which tends to 0 as $n \to \infty$.
d.)Suppose $\int |f - f_n| \to 0$ as $n \to \infty$.We have $||f_n| - |f|| \le |f_n - f|$ and this immediately gives $\int |f_n| \to \int |f|$ as $n \to \infty$.
Conversely, suppose that $\int |f_n| \to \int |f|$ as $n \to \infty$.We will use part $b.$).Take $g_n = 2(|f_n| + |f|)$ and note that $g_n \to 4|f|$ and note also that $|f|$ is integrable by Fatou's lemma.Now the result follows from part $b.$) by taking $f_n = |f_n - f| + |f_n| - |f|$.

PROBLEM 26. Evaluate

$$\lim_{n\to\infty} \int_0^n (1 + x/n)^n e^{-2x} dx,$$

justifying any interchange of limits you use.

SOLUTION. W know that $\lim_{n\to\infty}(1+x/n)^n = e^x$ and $(1+x/n)^n \le (1+x/(n+1))^{n+1}$. Also we have $(1 + x/n)^n \le e^x$. Therefore we get $(1 + x/n)^n \nearrow e^x$, which gives that

$(1+x/n)^n e^{-2x} \leq e^{-x}$.Therefore we can apply the Dominated Convergence Theorem to the function $(1+x/n)^n e^x$ with the dominating function e^{-x}.An easy computation gives,

$$\lim_{n\to\infty}\int_0^n (1+x/n)^n e^{-2x}dx = \lim_{n\to\infty}\int_0^\infty 1_{[0,n]}(x)(1+x/n)^n e^{-2x}dx$$
$$= \int_0^\infty \lim_{n\to\infty} 1_{[0,n]}(x)(1+x/n)^n e^{-2x}dx = \int_0^\infty e^{-x}dx = 1.$$

PROBLEM 27.
a.) Let (a_n) be a sequence of nonnegative real numbers. Set $\mu(\emptyset) = 0$, and for every nonempty subset A of N (set of natural numbers) set $\mu(A) = \sum_{n\in A} a_n$.Show that the set function $\mu : P(N) \to [0,\infty]$ is a measure.
b.) Let X be a nonempty set and let $f : X \to [0,\infty]$ be a function.Define μ by $\mu(A) = \sum_{a\in A} f(x)$ if $A \neq \emptyset$ and is at most countable, $\mu(A) = \infty$ if A is uncountable, and $\mu(\emptyset) = 0$.Show that μ is a measure.

SOLUTION.
a.) If (A_n) is a sequence of pairwise disjoint subsets of N and $A = \cup_{n=1}^\infty A_n$, then note that

$$\mu(A) = \sum_{k\in A_n} a_k = \sum_{n=1}^\infty (\sum_{k\in A_n} a_k) = \sum_{n=1}^\infty \mu(A_n).$$

This clearly shows that μ is a measure.
b.) We need to show the $\sigma-$additivity of μ. For that let (A_n) be asequence of pairwise disjoint sequence of subsets of X. Set $A = \cup_{n=1}^\infty A_n$. If some A_n is uncountable then A is likewise uncountable , and hence , in this case $\mu(A) = \sum_{n=1}^\infty \mu(A_n) = \infty$ holds. On the other hand, if each A_n is at most countable then A is also at most countable, and so

$$\mu(A) = \sum_{x\in A} f(x) = \sum_{n=1}^\infty [\sum_{a\in A_n} f(x)] = \sum_{n=1}^\infty \mu(A_n)$$

also holds.Therefore μ is $\sigma-$additive and hence it is a measure.

PROBLEM 28. Let F be a nonempty collection of subsets of a set X and let $f : F \to [0,\infty]$ be a function. Define $\mu : P(X) \to [0,\infty]$ by $\mu(\emptyset) = 0$ and

$$\mu(A) = \inf\{\sum_{n=1}^\infty f(A_n) : (A_n) \subseteq F, and, A \subseteq \cup_{n=1}^\infty A_n\}$$

for each $A \neq \emptyset$, with $\inf \emptyset = \infty$. Show that μ is an outer measure.

SOLUTION.(1) By definition we have $\mu(\emptyset) = 0$.
(2) (Monotonicity) : Let $A \subseteq B$ and let (A_n) be a sequence in F with $B \subseteq \cup_{n=1}^\infty A_n$. Then $A \subseteq \cup_{n=1}^\infty A_n$, and so $\mu(A) \leq \sum_{n=1}^\infty f(A_n)$. Therefore

$$\mu(A) \leq \inf\{\sum_{n=1}^\infty f(A_n) : (A_n) \subseteq F, and, B \subseteq \cup_{n=1}^\infty A_n\} = \mu(B).$$

If there is no sequence (A_n) with $B \subseteq \cup_{n=1}^{\infty} A_n$, then $\mu(B) = \infty$, and clearly $\mu(A) \leq \mu(B)$.
(3) (Subadditivity) : Let (E_n) be a sequence of subsets of X and let $E = \cup_{n=1}^{\infty} E_n$.If $\sum_{n=1}^{\infty} \mu(E_n) = \infty$, then $\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$ is obvously ture. So, assume $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ and let $\epsilon > 0$. For each n pick a sequence (A_k^n) of F with $E_n \subseteq \cup_{k=1}^{\infty} A_k^n$ and

$$\sum_{k=1}^{\infty} f(A_k^n) < \mu(E_n) + \epsilon/2^n.$$

Clearly, $E \subseteq \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} A_k^n$ holds, and so

$$\mu(E) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f(A_k^n) < \sum_{n=1}^{\infty} [\mu(E_n) + \epsilon/2^n] = \sum_{n=1}^{\infty} \mu(E_n) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary , it follows that

$$\mu(E) = \mu(\cup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Therefore μ is an outermeasure.

PROBLEM 29. Let $f : R \to R$ be a Lebesgue integrable function.Show that

$$\lim_{t\to\infty} \int f(x)cos(xt)d\lambda(x) = \lim_{t\to\infty} \int f(x)sin(xt)d\lambda(x) = 0$$

SOLUTION. This is known as the Riemann-Lebesgue lemma. Since simple functions are dense in integrable functions, it suffices to prove the result for the special case $f = 1_{[a,b)}$ where 1_B represents the characteristic function of the set B. So, let $f = 1_{[a,b)}$ where $-\infty < a < b < \infty$. In this case, for each $t > 0$ we have

$$\left| \int f(x)cos(xt)d\lambda(x)\right| = \left| \int_a^b cos(xt)dx\right| = |sin(xt)/t|_{x=a}^{x=b}| = |\{sin(bt) - sin(at)\}/t| \leq 2/t,$$

and so $\lim_{t\to\infty} \int f(x)cos(xt)d\lambda(x) = 0$ holds. In a similar fashion, we can show that $\lim_{t\to\infty} \int f(x)sin(xt)d\lambda(x) = 0$.

PROBLEM 30. For a sequence (A_n) of subsets of a set X define
$\liminf A_n = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$ and $\limsup A_n = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i$
Now let (X, S, μ) be a measure space and let (E_n) be a sequence of measurable sets.Show the following:
a.) $\mu(\liminf E_n) \leq \liminf \mu(E_n)$
b.) If $\mu(\cup_{n=1}^{\infty} E_n) < \infty$, then $\mu(\limsup E_n) \geq \liminf \mu(E_n)$

SOLUTION.
a.) Note that $\cap_{i=n}^{\infty} E_i \nearrow \liminf E_n$ and $\cap_{i=n}^{\infty} \subseteq E_n$ holds for each n.Thus,

$$\mu(\liminf E_n) = \lim_{n\to\infty} \mu(\cap_{i=n}^{\infty} E_i) \leq \liminf \mu(E_n).$$

b.) Note that $\cup_{i=n}^{\infty} E_i \searrow \limsup E_n$. Hence,since $\mu(\cup_{n=1}^{\infty} E_n) < \infty$

$$\mu(\limsup E_n) = \lim_{n \to \infty} \mu(\cup_{i=n}^{\infty} E_i) \geq \limsup \mu(E_n).$$

PROBLEM 31.
a.) Let X be a nonempty set and let δ be the Dirac measure on X with respect to a point. Show that every function $f : X \to R$ is integrable and that $\int f d\delta = f(a)\delta(a) = f(a)$.
b.) Let μ be the counting measure on N(set of natural numbers). Show that a function $f : N \to R$ is integrable if and only if $\sum_{n=1}^{\infty} |f(n)| < \infty$. Also, show that in this case $\int f d\mu = \sum_{n=1}^{\infty} f(n)$.

SOLUTION.
a.) Note that $f = f(a)1_{\{a\}}$ a.e. holds. Therefore, the function f is integrable and $\int f d\delta = f(a)\delta(\{a\}) = f(a)$.
b.) Let $f : N \to R$. Since every function is measurable, f is integrable if and only if both f^+ and f^- are integrable. So, we can assume that $f(k) \geq 0$ holds for each k.
If $\phi_n = \sum_{k=1}^{n} f(k)1_{\{k\}}$, then (ϕ_n) is a sequence of step functions such that $\phi_n \nearrow f(k)$ as $n \to \infty$ for each k, and

$$\int \phi_n d\mu = \sum_{k=1}^{n} f(k) \nearrow \sum_{k=1}^{\infty} f(k)$$

as $n \to \infty$. This shows that f is integrable if and only if $\sum_{k=1}^{\infty} f(k) < \infty$, and in this case $\int f g\mu = \sum_{k=1}^{\infty} f(k)$.

PROBLEM 32. Let (X, S, μ) be a measure space and let $f_1, f_2, f_3, ...$ be nonnegative integrable functions such that $f_n \to f$ a.e. and $\lim \int f_n d\mu = \int f d\mu$. If E is a measurable set, then show that $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$.

SOLUTION. By assumptions the functions $f_1 1_E, f_2 1_E, f_3 1_E, ...$are nonnegative and integrable (because $0 \leq f_n 1_E \leq f_n$) and $f_n 1_E \to f 1_E$ holds. Using Fatou's lemma we get

$$\int_E f d\mu = \int \liminf f_n 1_E d\mu \leq \liminf \int f_n 1_E d\mu = \liminf \int_E f_n d\mu.(*)$$

Similarly, we have

$$\int_{E^c} f d\mu \leq \liminf \int_{E^c} f_n d\mu.(**)$$

Therefore,

$$\int f d\mu = \int_E f d\mu + \int_{E^c} f d\mu \leq \liminf \int_E f_n d\mu + \liminf \int_{E^c} f_n d\mu$$

$$\leq \liminf (\int_E f_n d\mu + \int_{E^c} f_n d\mu) = \liminf (\int f_n d\mu) = \int f d\mu,$$

It follows that

$$\int_E f d\mu + \int_{E^c} f d\mu = \liminf \int_E f_n d\mu + \liminf \int_{E^c} f_n d\mu,$$

and from (*) and (**), we see that

$$\liminf \int_E f_n d\mu = \int_E f d\mu.$$

Now let (g_n) be a subsequence of (f_n). Then ,
$g_n \to f$ a.e. and $\lim_{n\to\infty} \int g_n d\mu = \int f d\mu$. By the above result, we infer that

$$\liminf \int_E g_n d\mu = \int_E f d\mu,$$

and so there exists a subsequence (g_{kn}) of the sequence (g_k) such that $\lim \int_E g_{kn} d\mu = \int_E f d\mu$. In other words, we have shown that every subsequence of a sequence of real numbers $(\int_E f_n d\mu)$ has a convergent subsequence converging to $\int_E f d\mu$.This means that $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$ holds.

PROBLEM 33. Let $f : [0, \infty) \to R$ be a continuous function such that $f(x+1) = f(x)$ holds for all $x \geq 0$. If $g : [0, 1] \to R$ is an arbitrary continuous function, then show that

$$\lim_{n\to\infty} \int_0^1 g(x)f(nx)dx = (\int_0^1 g(x)dx)(\int_0^1 f(x)dx).$$

SOLUTION. Observe that by induction $f(x + k) = f(x)$ holds for all $x \geq 0$ and for all nonnegative integers k.
The change of variables $u = nx$ yields

$$\int_0^1 g(x)f(nx)dx = 1/n \int_0^n g(u/n)f(u)du = 1/n \sum_{i=1}^{n} \int_{i-1}^{i} g(u/n)f(u)du.$$

Letting $t = u - i + 1$, we get

$$\int_{i-1}^{i} g(u/n)f(u)du = \int_0^1 g((t + i - 1)/n)f(t + i - 1)dt = \int_0^1 g((t + i - 1)/n)f(t)dt.$$

Consequently,

$$\int_0^1 g(x)f(nx)dx = \int_0^1 [\sum_{i=1}^{n} 1/ng((t + i - 1)/n)]f(t)dt = \int_0^1 h_n(t)dt(*)$$

where $h_n(t) = [\sum_{i=1}^{n} 1/ng((t + i - 1)/n)]f(t)$. Clearly, h_n is a continuous function defined on $[0, 1]$. In addition, note that if $|g(x)| \leq K$ and $|f(x)| \leq K$ hold for each

$x \in [0,1]$, then $h_n(t) \leq K^2$ for all $t \in [0,1]$.i.e. the sequence (h_n) is uniformly bounded on $[0,1]$. Now, note that if $0 \leq t \leq 1$ then $(i-1)/n \leq (t+i-1)/n \leq i/n$.Thus, if m_i^n and M_i^n denote the minimum and maximum values of g, respectively, on the interval $[(i-1)/n, i/n]$, then $m_i^n \leq g((t+i-1)/n) \leq M_i^n$ holds for each $0 \leq t \leq 1$. Let
$R_n = \sum_{i=1}^n 1/nm_i^n$ and $S_n = \sum_{i=1}^n 1/nM_i^n$,
and note that R_n and S_n are two Riemann sums(the smallest and the largest ones) for the function g corresponding to the partition $\{0, 1/n, 2/n, ..., (n-1)/n, 1\}$. Hence, $\lim_{n\to\infty} R_n = \lim_{n\to\infty} S_n = \int_0^1 g(x)dx$.From,

$$|h_n(t) - R_n.f(t)| = |[\sum_{i=1}^n 1/ng((t+i-1)/n)]f(t) - R_n.f(t)|$$

$$= |([\sum_{i=1}^n 1/ng((t+i-1)/n)] - R_n).f(t)| \leq (S_n - R_n)|f(t)|,$$

we see that $\lim_{n\to\infty} h_n(t) = f(t)\int_0^1 g(x)dx$ and in fact h_n converges uniformly. Now, by (*) and the Lebesgue Dominated Convergence Theorem we obtain

$$\lim_{n\to\infty} \int_0^1 g(x)f(nx)dx = \lim_{n\to\infty} \int_0^1 h_n(t)dt$$

$$= \int_0^1 [\lim_{n\to\infty} h_n(t)]dt = \int_0^1 [f(t)\int_0^1 g(x)dx]dt = (\int_0^1 g(x)dx)(\int_0^1 f(x)dx).$$

PROBLEM 34. Show that $\int_0^\infty \frac{sin^2(x)}{x^2}dx = \frac{\pi}{2}$.

SOLUTION. Consider the function $f(x) = 1, if x = 0, f(x) = sin^2(x)/x^2, if 0 < x \leq 1, f(x) = 1/x^2, if x > 1$.Note that f is Lebesgue integrable over $[0,\infty)$.By the inequality $0 \leq \frac{sin^2(x)}{x^2} \leq f(x)$, we see that the function $\frac{sin^2(x)}{x^2}$ is Lebesgue integrable over $[0,\infty)$. Now for each $r, \epsilon > 0$, we have

$$\int_\epsilon^r \frac{sin^2(x)}{x^2}dx = -\int_\epsilon^r sin^2(x)d(\frac{1}{x})$$

$$= \frac{-sin^2(x)}{x}|_\epsilon^r + \int_\epsilon^r 2sinx\frac{cosx}{x}dx$$

$$= \frac{sin^2\epsilon}{\epsilon} - \frac{sin^2 r}{r} + \int_{2\epsilon}^{2r} \frac{sinx}{x}dx.$$

Thus, we see that

$$\int_0^\infty \frac{sin(x)}{x^2}dx = \lim_{\substack{r\to\infty \\ \epsilon\to 0^+}} \int_\epsilon^r \frac{sin^2(x)}{x^2}dx = \int_0^\infty \frac{sin(x)}{x}dx = \frac{\pi}{2}.$$

PROBLEM 35.
a.)Let (f_n) be a sequence of measurable functions and let $f : X \to R$. Assume that

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0(*)$$

holds for every $\epsilon > 0$. Show that f is measurable.
b.) Assume that $(f_n) \subseteq M$ satisfies $f_n \uparrow$ and $f_n \to^{\mu} f$(i.e. f_n goes to f in measure). Show that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

c.) Assume that $(f_n) \subseteq M$ satisfies $f_n \geq 0$ a.e. and $f_n \to^{\mu} f$(i.e. f_n goes to f in measure). Show that $f \geq 0$ a.e.

SOLUTION.
a.)We will show that there is a subsequence (f_{nk}) of (f_n) that converges to f a.e. as the given condition $(*)$ is equivalent to the condition that $f_n \to f$ in measure.
Pick a sequence (k_n) of strictly increasing positive integers such that $\mu(\{x : |f_n(x) - f(x)| \geq 1/n\}) < 2^{-n}$ for all $k > k_n$. Set $E_n := \{x : |f_n(x) - f(x)| \geq 1/n\}$ for each n and let $E := \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$.Then,

$$\mu(E) \leq \mu(\cup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} \mu(E_k) \leq 2^{1-n}$$

holds for all n, and this shows that $\mu(E) = 0$. Also, if x is not in E, then there exists some n such that x is not in $\cup_{k=n}^{\infty} E_k$, and so $|f_{km} - f| \leq 1/m$ holds for each $m \geq n$.Therefore, $\lim f_{kn}(x) = f(x)$ for each $x \in E^c$, and so $f_{kn} \to f$ a.e. holds. Thus, f is measurable as limit of a sequence of measurable functions is itself measurable.
b.)By part a.) there exists a subsequence f_{nk} which converges to f a.e. Since $f_n \uparrow$, it easily follows that $f_n \uparrow f$.Now apply the Monotone Convergence Theorem to deduce the result.
c.) Again by part a.) there exists a subsequence f_{nk} which converges to f a.e. Thus, we must have $f \geq 0$ a.e.

PROBLEM 36. Let g be an integrable function and let (f_n) be a sequence of integrable functions such that $|f_n| \leq g$ a.e. holds for all n. Suppose that $f_n \to^{\mu} f$ (i.e.f_n goes to f in measure), then show that f is an integrable function and $\lim \int |f_n - f| d\mu = 0$.

SOLUTION. By the above exercise we know that if f_n converges to f in measure then there exists a subsequence f_{nk} of f_n which converges to f a.e.Thus, $|f| \leq g$ a.e. And application of the Lebesgue Dminated Convergence Theorem gives that f is integrable. Now, assume on the contrary that $\lim \int |f_n - f| d\mu \neq 0$.Thus, assume that for some $\epsilon > 0$ there exists a subsequence (g_n) of (f_n) such that $\int |g_n - f| d\mu \geq \epsilon$. But we know that there exists a subsequence (h_n) of (g_n) with $h_n \to f$ a.e. Now Lebesgue Dominated

Convergence Theorem implies) $< \epsilon \leq \int |h_n - f| d\mu \to 0$, which is a contradiction. Therefore, we must have $\lim \int |f_n - f| d\mu = 0$.

PROBLEM 37. Let f be a.e. positive measurable function and let

$$m_i = \mu(\{x \in X : 2^{i-1} < f(x) \leq 2^i\})$$

for each integer i. Show that f is integrable if and only if $\sum_{-\infty}^{\infty} 2^i m_i < \infty$.

SOLUTION. Let $E_i := \{x \in X : 2^{i-1} < f(x) \leq 2^i\}, i = 0, +^-1, +^-2, ...$Set $\phi_n = \sum_{i=-n}^{n} 2^i 1_{E_i}$ for $n = 1, 2, 3, ...$ Then there exists some function g with $\phi_n \uparrow g$ a.e.Clearly, g is a measurable function and $0 \leq f \leq g$ a.e.
Assume that f is integrable. Then, each ϕ_n is a step function, and by $\phi_n \leq 2f$, it follows that

$$\sum_{i=-\infty}^{\infty} 2^i m_i = \lim_{n \to \infty} \int \phi_n d\mu \leq 2 \int f d\mu < \infty$$

On the other hand, if $\sum_{-\infty}^{\infty} 2^i m_i < \infty$, then each ϕ_n is a step function, and so g is integrable. Since $0 \leq f \leq g, f$ it follows that f is also integrable.

PROBLEM 38.
a.) Let $f \in L_1(\mu)$(i.e. f is integrable) and let $\epsilon > 0$. Show that

$$\mu(\{x \in X : |f(x)| \geq \epsilon\}) \leq \epsilon^{-1} \int |f| d\mu$$

b.) If $f_n \to f$ in $L_1(\mu)$ then show that $f_n \to f$ in measure.

SOLUTION.
a.) Consider the measurable set $E = \{x \in X : |f(x)| \geq \epsilon\}$.Then,the follwing estimate gives the result.

$$\int |f| d\mu \geq \int |f| 1_E d\mu \geq \int \epsilon 1_E d\mu = \epsilon \mu(E).$$

b.) From part a.) we have the follwing inequality,

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \leq \epsilon^{-1} \int |f_n - f| d\mu.$$

But if $f_n \to f$ in $L_1(\mu)$ holds then the right hand side of this inequality goes to 0 and that shows that $f_n \to f$ in measure.

PROBLEM 39. Suppose f is integrable on a set A. Then, show that given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$| \int_E f(x) d\mu | < \epsilon$$

for every measurable set $E \subset A$ of measure less than δ.

SOLUTION. There are various ways to do this. Here is our proof :
The result is obvious when f is bounded, since then

$$|\int_E f(x)dx| \leq \int_E |f|d\mu \leq (sup_{x\in E}|f(x)|)\mu(E)$$

In the general case, let

$$A_n := \{x \in A : n \leq f(x) \leq n+1\}$$

$$B_N := \cup_{n=0}^{N} A_n$$

$$C_N := A - B_N$$

Then, $\int_A |f(x)|dx = \sum_{n=0}^{\infty} \int_{A_n} |f(x)|d\mu$.Let N be such that

$$\sum_{n=N+1}^{\infty} \int_{A_n} |f(x)|d\mu = \int_{C_N} |f(x)|d\mu < \frac{\epsilon}{2}.$$

and let $0 < \delta < \frac{\epsilon}{2(N+1)}$. Then $\mu(E) < \delta$ implies

$$|\int_E f(x)d\mu| = \int_E |f(x)|d\mu = \int_{E\cap B_N} |f(x)|d\mu + \int_{E\cap C_N} |f(x)|d\mu$$

$$\leq (N+1)\mu(E) + \int_{C_N} |f(x)d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

PROBLEM 40. Suppose f is integrable function on $R(\Leftrightarrow f \in L^1(R))$.Then, show that

$$\lim_{t\to 0} \int |f(t+x) - f(x)|dx = 0.$$

SOLUTION. Note that the result is immediate when f is a step function. Now let f be an arbitrary integrable function and let $\epsilon > 0$. If $f \approx f_1 + f_2 + f_3 + ...$(approximate f by a step function which can be done as step functions are dense in L^1), then there exists $n_0 \in N$ such that

$$\sum_{n=n_0+1}^{\infty} \int |f_n|dx < \frac{\epsilon}{3}.$$

We have

$$\int |f(x+t) - f(x)|dx \leq \int |\sum_{n=1}^{n_0} f_n(x+t) - \sum_{n=1}^{n_0} f_n(x)|dx$$

$$+ \sum_{n=n_0+1}^{\infty} \int |f_n(x+t)|dx + \sum_{n=n_0+1}^{\infty} \int |f_n(x)|dx$$

$$= \int |\sum_{n=1}^{n_0} f_n(x+t) - \sum_{n=1}^{n_0} f_n(x)|dx$$

$$+2 \sum_{n=n_0+1}^{\infty} \int |f_n(x)|dx$$

$$< \int |\sum_{n=1}^{n_0} f_n(x+t) - \sum_{n=1}^{n_0} f_n(x)|dx + 2\frac{\epsilon}{3}$$

Since $\sum_{n=1}^{n_0} f_n(x)$ is a step function, we have

$$\lim_{t\to 0} \int |\sum_{n=1}^{n_0} f_n(x+t) - \sum_{n=1}^{n_0} f_n(x)|dx = 0.$$

Consequently, $\int |f(t+x) - f(x)|dx < \epsilon$ for sufficiently small t.This proves the result.

PROBLEM 41. Show that every extended real valued measurable function f is the limit of a sequence (f_n) of simple functions.

SOLUTION. Suppose first that $f \geq 0$. For every $n = 1, 2, 3, ...$, and for every $x \in X$, write
$f_n(x) = (i-1)/2$ if $(i-1)/2 \leq f(x) < i/2$, for $i = 1, 2, ...2^n n$
$f_n(x) = n$ if $f(x) \geq n$.
Clearly f_n is a nonnegative simple function, and the sequence (f_n) is increasing. If $f(x) < \infty$, then, for some n,

$$0 \leq f(x) - f_n(x) \leq 2^{-n};$$

if $f(x) = \infty$, then $f_n(x) = n$ for every n.Recalling that the difference of two simple functions is a simple function, application of the procedure above to f^+ and f^- separately proves the result for arbitrary f.

PROBLEM 42. Suppose μ is a probability measure on X $i.e.\mu(X) = 1$.Let $A_1, A_2, A_3, ... \in U$ be sets in the $\sigma-$algebra U such that $\sum_{i=1}^{n} \mu(A_i) > n-1$. Show that $\mu(\cap_{k=1}^{n} A_k) > 0$.

SOLUTION. Since $\mu(A_i^c) = 1 - \mu(A_i)$, we have $\sum_{i=1}^{n} \mu(A_i^c) = n - \sum_{i=1}^{n} \mu(A_i) < 1$.By the semi-additivity of the measure we have,

$$\mu(\bigcup_{i=1}^{n} A_i^c) \leq \sum_{i=1}^{n} \mu(A_i^c) < 1.$$

Therefore,

$$\mu(\bigcap_{i=1}^{n} A_i) = 1 - \mu((\bigcap_{i=1}^{n} A_i)^c) = 1 - \mu(\bigcup_{i=1}^{n} A_i^c) > 0.$$

PROBLEM 43. Suppose f is an integrable function on $X = R^p$.
i) Show that $\forall\epsilon > 0$, there exists a measurable set with finite measure such that f is bounded on A and $\int_{(X-A)} |f| d\mu < \epsilon$
ii) From this deduce that

$$\lim_{\mu(E)\to 0} \int_E |f| d\mu = 0.$$

SOLUTION.
i)We can assume that $f \geq 0$. Consider the following sets

$$A_0 := \{x : f(x) = 0\}, A_n := \{x : 1/n \leq f(x) \leq n\}, A_\infty := \{x : f(x) = \infty\}.$$

Clearly, $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ and also $\bigcap_{n=1}^{\infty}(X - A_n) = A_0 \cup A_\infty$. Note that $\mu(A_\infty) = 0$ as f is integragle. Thus, we have

$$\lim_{n\to\infty} \int_{(X-A_n)} f d\mu = \int_{A_0} f d\mu + \int_{A_\infty} f d\mu = 0$$

Therefore $\exists n_0$ such that letting $A = A_{n_0}$, $\int_{(X-A)} f d\mu < \epsilon$. Furthermore, f is bounded on A by n_0 and also A has finite measure as $\frac{1}{n_0}\mu(A) \leq \int_A f d\mu < \infty$
ii) Let A be as in part i) with $\frac{\epsilon}{2}$ instead of ϵ. Let $M = \sup_A f$. Then if E is a measurable set and $\mu(E) < \frac{\epsilon}{2M}$ we have

$$\int_E f d\mu \leq \int_{(X-A)} f d\mu + \int_{(E\cap A)} f d\mu$$

$$\leq \int_{(X-A)} f d\mu + \int_{(E\cap A)} f d\mu < \frac{\epsilon}{2} + M\mu(E) < \frac{\epsilon}{2} + M\frac{\epsilon}{2} = \epsilon.$$

PROBLEM 44.
i) Show that an algebra A is a $\sigma-$algebra if and only if it is closed under countable increasing unions(i.e. If $(E_j)_{j=1}^{\infty} \subset A$ and $E_1 \subset E_2 \subset E_3 \subset ...$, then $\bigcup_{j=1}^{\infty} E_j \in A$.)
ii) Suppose $\mu_1, \mu_2, \mu_3, ...$ are measures on (X, M) and $a_1, a_2, a_3, ... \in [0, \infty)$. Show that $\sum_1^n a_j\mu_j$ is a measure on (X, M).

SOLUTION.
i) We just need to show that A is closed under all countable unions since the other direction is obvious. For that, let (E_j) be any sequence of sets in A. Let $F_1 = E_1$ and $F_n = \cup_{j=1}^{n} E_j$ for $n > 1$. Since A is an algebra, $F_n \in A$. Note that $F_n \subseteq F_{n+1}$, so

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} F_j \in A.$$

ii) Let $m(E) = \sum_1^n a_j\mu_j(E)$. It is clear that $m(\emptyset) = 0$ as $m(\emptyset) = \sum_1^n a_j\mu_j(\emptyset) = 0$.It is also obvious that $m(E) \geq 0$.For the countable additivity, recall the fact that series of nonnegative terms can be added and multiplied termwise and rearrenged arbitrarily without changing the sum. Thus, m is a measure.

PROBLEM 45.
i) Let (X, M, μ) be a measure space and $E, F \in M$. Show that

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

ii) Given a measure space (X, M, μ) and $E \in M$, define $\mu_E(F) = \mu(A \cap E)$. Show that μ_E is a measure.

SOLUTION.
i) Write the set as a disjoint union of two sets as follows $E = (E - F) \cup (E \cap F)$.Now write F as the disjoint union $F = (F - E) \cup (E \cap F)$ and $E \cup F$ as the disjoint union $(E - F) \cup (F - E) \cup (E \cap F)$. Then we have,

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E - F) + \mu(F - E) + 2\mu(E \cap F)$$

$$= \mu(E - F) + \mu(E \cap F) + \mu(F - E) + \mu(E \cap F)$$

$$= \mu(E) + \mu(F)$$

ii) Clearly, $\mu_E(A) \geq 0$ and $\mu_E(\emptyset) = 0$. Countable additivity is also very easy to verify. Suppose $A_n \cap A_m = \emptyset$ when $m \neq n$, then

$$\mu_E(\bigcup_{n=1}^{\infty} A_n) = \mu(E \cap \bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(E \cap A_n) = \sum_{n=1}^{\infty} \mu_E(A_n).$$

PROBLEM 46. We know that if μ is a measure(in the sense that it is countably additive) then it is continuous from below. {i.e. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq$ ...then $\mu(\bigcup A_i) = \lim \mu(A_i)$}.Similarly, we know that if $\mu(X)$ is finite then μ is continuous from above.{i.e. If ... $\subseteq A_3 \subseteq A_2 \subseteq A_1$ then $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap A_n)$.}. Show that a finitely additive measure is a measure if and only if it is continuous from below.Now suppose that $\mu(X) < \infty$. Show that μ is a measure if and only if it is continuous from above.Give a counterexample to show that if μ is a measure with $\mu(X) = \infty$ then it is not necessarily continuous from above.

SOLUTION. Suppose that μ is a finitely additive and continuous from below. Let $(E_j)_{j=1}^{\infty}$ be a disjoint sequence of measurable sets. Then for each n let $F_n = \cup_{j=1}^{n} E_j$, so that we have $F_1 \subseteq F_2 \subseteq$ .... By finite additivity, $\mu(F_n) = \sum_{j=1}^{n} \mu(E_j)$, and by continuity from below,

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \mu(\bigcup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n) = \lim \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j).$$

This clearly shows that μ is countably additive, and hence is a measure.
We will try to use previous part. Suppose $\mu(X) < \infty$ and μ is continuous from above. We will show that under these conditions μ is also continuous from below which, together with previous part, will prove that μ is countably additive and hence is a measure. Take a decresing sequence of measurable sets $E_1 \subseteq E_2 \subseteq ...$, then $... \subseteq E_2^c \subseteq E_1^c$, and so,

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \mu(X - \bigcap_{j=1}^{\infty} E_j^c) = \mu(X) - \mu(\bigcap_{j=1}^{\infty} E_j^c)$$

$$= \mu(X) - \lim_{j\to\infty} \mu(E_j^c) = \lim_{j\to\infty} [\mu(X) - \mu(E_j^c)] = \lim_{j\to\infty} \mu(E_j).$$

Hence μ is continuous from below and by the previous part it is therefore countaly additive.

PROBLEM 47. Let μ^* be an outer measure on X and let $(A_j)_{j=1}^{\infty}$ be a sequence of disjoint μ^*- measurable sets(in the sense of Carethedory). Show that $\mu^*(E \cap (\bigcup_1^{\infty} A_j)) = \sum_1^{\infty} \mu^*(E \cap A_j)$ for any $E \subset X$.

SOLUTION. First we prove that for each finite n,

$$\mu^*(E \cap \bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu^*(E \cap A_j)$$

For this we use induction on n. This statement is obvious when $n = 1$. Suppose that it is true for $n = k$.Let $n = k + 1$ and note that

$$\mu^*(E \cap \bigcup_{j=1}^{k+1} A_j) \geq \mu^*\big((E \cap \bigcup_{j=1}^{k+1} A_j) \cap A_{k+1}\big) + \mu^*\big((E \cap \bigcup_{j=1}^{k+1} A_j) \cap A_{k+1}^c\big).$$

$$= \mu^*(E \cap A_{k+1}) + \mu^*(E \cap \bigcup_{j=1}^{k} A_j) = \mu^*(E \cap A_{k+1}) + \sum_{j=1}^{k} \mu^*(E \cap A_j)$$

By the monotonicity of the outer measure , we then have,

$$\mu^*(E \cap \bigcup_{j=1}^{\infty} A_j) \geq \mu^*(E \cap \bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu^*(E \cap A_j).$$

Since n is arbitrary, it follows that

$$\mu^*(E \cap \bigcup_{j=1}^{\infty} A_j) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$$

The other inequality follows from the countable subadditivity.Hence equality holds.

PROBLEM 48. Let $f : R^n \to R$ be a Lebesgue measurable function such that

$$m(\{x : |f(x)| > t\}) \le \frac{c}{t^2}, t > 0.$$

Prove that there exists a constant C_1 such that for any Borel set $E \subset R^n$ of finite and positive measure

$$\int_E |f(x)|dx \le C_1 \sqrt{m(E)}.$$

SOLUTION.

$$\int_E |f|dm = \int_0^\infty m(\{x : |f| > t\} \cap E)dt$$

$$= \int_0^{\sqrt{\frac{c}{|E|}}} m(\{x : |f| > t\} \cap E)dt + \int_{\sqrt{\frac{c}{|E|}}}^\infty m(\{x : |f| > t\} \cap E)dt$$

$$\le |E|\sqrt{\frac{c}{|E|}} + \int_{\sqrt{\frac{c}{|E|}}}^\infty \frac{c}{t^2}dt = \sqrt{c}\sqrt{|E|} + [\frac{-c}{t}]_{\sqrt{\frac{c}{|E|}}}^\infty$$

$$= \sqrt{c}\sqrt{|E|} + \frac{c}{\sqrt{\frac{c}{|E|}}} = 2\sqrt{c}\sqrt{|E|} = C_1\sqrt{|E|}.$$

Thus $C_1 = 2\sqrt{c}$ and therefore the assertion is proved.

PROBLEM 49. Let $B(m, 1)$ be $m-$dimensional ball of radius 1 centered at the origin in R^m.
a.) Show that there exists a function $f : R \to [0, 1]$ such that

$$m(B(n + 1, 1)) = m(B(n, 1)) \int [f(t)]^n dt.$$

Here m denotes the Lebesgue measure.
b.) Show that $\int [f(t)]^n dt \to 0$ as $n \to \infty$.
c.) Show that for any positive number A, $A^n m(B(n, 1)) \to 0$ as $n \to \infty$.

SOLUTION.
a.) Let $B(n+1, r) = \{x \in R^n : x_1^2 + ...x_{n+1}^2 = r\}$.Integrate over $t = x_{n+1}$ to determine the volume of $B(n + 1, 1)$. Then we get,

$$m(B(n + 1, 1)) = \int_{-1}^1 m(B(n, \sqrt{1 - t^2}))dt$$

$$= 2\int_{-1}^1 m(B(n, 1))\left(\sqrt{1 - t^2}\right)^n dt = m(B(n, 1)) \int_{-\infty}^\infty f(t)dt$$

where $f(t) = 1_{[-1,1]}\sqrt{1 - t^2}$, here 1_A represents the characteristic function of the set A. The first equality follows from the $Fubini's$ theorem, and the second equality follows

from the fact that $m(B(n,r)) = r^n m(B(n,1))$. This fcat follows from a linear change of variables.
b.) Note that $[f(t)]^n \to 0$ pointwise if $t \neq 0$ and define $f_n := f^n$, then $f_1 \geq f_2 \geq ... \geq 0$, f_1 is integrable and $f_n \to 0$ almost everywhere, so by the first question we have $\int [f(t)]^n dt \to 0$.
c.)

$$A^n m(B(n,1)) = A^n m(B(n-1,1) \int [f(t)^{n-1} dt$$

$$= A^n m(B(n-2,1)) \int [f(t)]^{n-2} dt \int [f(t)]^{n-1} dt$$

$$= ... = A^n m(B(0,1)) \int [f(t)]^0 dt ... \int [f(t)]^{n-1} dt$$

$$\left(A \int [f(t)]^0 dt \right) ... \left(A \int [f(t)]^{n-1} dt \right).$$

Here we used the fact that $m(B(0,1)) = 1$. By part $b.$), for k sufficiently large, $A \int [f(t)]^k dt < s < 1$ for some fixes s with $0 \leq s < 1$. But this shows $A^n m(B(n,1)) \searrow 0$.

PROBLEM 50. Let f be an integrable function. Show that
i) $\mu(\{x : |f(x)| \geq a\}) \leq \frac{1}{a} \int |f| d\mu$.
ii) $\mu(\{x : |f(x)| \geq a\}) = o(\frac{1}{a})$ as $a \to \infty$.

SOLUTION.
i)

$$\mu(\{x : |f(x)| \geq a\}) = \int_{\{x:|f(x)|\geq a\}} d\mu \leq \int_{\{x:|f(x)|\geq a\}} \frac{|f|}{a} d\mu \leq \frac{1}{a} \int |f| d\mu.$$

ii) From part i) if $a \to \infty$ then $\mu(\{x : |f(x)| \geq a\}) \to 0$. This clearly proves the result.

PROBLEM 51. Let (X, M) be a measure space. If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$, for $E \in M$. Show that λ is a measure on M, and for any $g \in L^+$, $\int g d\lambda = \int f g d\mu$.

SOLUTION. Since $f \geq 0$, $\lambda(E) = \int_E f d\mu \geq 0$, and $\lambda(\emptyset) = 0$. If A is a disjoint union of $(A_n)_{n=1}^{\infty}$, then

$$\lambda(A) = \int_A f d\mu = \int 1_A . f d\mu = \int \sum_n 1_{A_n} . f d\mu = \sum_n \int 1_{A_n} . f d\mu = \sum_n \int_{A_n} f d\mu = \sum_n \lambda(A_n).$$

Therefore, λ is a measure on M.
If $g \in L^+$ is simple, and $g = \sum_1^n a_k . 1_{E_k}$, then

$$\int g d\lambda = \sum_1^n a_k . \lambda(E_k) = \sum_1^n a_k . \int_{E_k} f d\mu = \int \sum_1^n a_k . 1_{E_k} . f d\mu = \int g f d\mu.$$

If $g \in L^+$ is arbitrary, we can find a sequence $(\phi_n)_1^\infty$ of nonnegative simple functions that increases pointwise to the function g. Then the sequence $\big(\phi_n.f\big)$ increases pointwise to the function $g.f$. Thus, by the Monotone Convergence Theorem,

$$\int g d\lambda = \lim_{n\to\infty} \int \phi_n d\lambda = \lim_{n\to\infty} \int \phi_n f d\mu = \int g f d\mu.$$

PROBLEM 52. Let $f(x) = x^{-1/2}$ if $0 < x < 1, f(x) = 0$ otherwise. Let $(r_n)_1^\infty$ be an enumeration of rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.Show that
a.) $g \in L^1(m)$ and in particular $g < \infty$ a.e.$\big($ Here m is the Lebesgue measure $\big)$.
b.) $g^2 < \infty$ a.e. but g^2 is not integrable on any interval.
c.) g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

SOLUTION.
a.) Let $f_n := 1_{[1/n,1]} f$, then $f_n \geq 0$ for all n and $f_n \nearrow f$ pintwise. We have,

$$\int f_n dm = \int_{1/n,1]} f dm = \int_{1/n}^1 x^{-1/2} dx = 2 - 2.(\frac{1}{n})^{1/2}.$$

Thus, since $f_n \nearrow f$ pointwise

$$\int f dm = \lim_{n\to\infty} \int f_n dm = \lim_{n\to\infty} 2 - 2.(\frac{1}{n})^{1/2} = 2.$$

Therefore, we have

$$\int |g| dm \leq \sum_{n=1}^\infty 2^{-n} \int |f(x - r_n)| dm = \sum_{n=1}^\infty 2^{-n}.2 = 2 < \infty.$$

and this clearly shows that $g \in L^1(m)$, and $g < \infty$ a.e.
b.) Since $g < \infty$ a.e., it is also true that $g^2 < \infty$ a.e. Fix any interval $\big(a, b\big)$ for some $a < b$, there is $r_n \in \big(a, b\big) \cap Q$ since Q is dense in R. There exists $M \in \mathbf{N}$ such that when $m \geq M$, $r_n + \frac{1}{m} \in (a, b)$. Then

$$1_{(a,b)} g^2(x) \geq 2^{-2n} 1_{[r_n + \frac{1}{m}, r_n + \frac{1}{M}]} f(x - r_n)^2$$

for $m \geq M$. Therefore,

$$\int_{(a,b)} g^2 dm \geq 2^{-2n} \int_{[r_n + \frac{1}{m}, r_n + \frac{1}{M}]} f(x - r_n)^2 dm$$

$$= 2^{-2n} \int_{[r_n + \frac{1}{m}}^{r_n + \frac{1}{M}]} (x - r_n)^{-1} dm = 2^{-2n} \Big(ln(m) - ln(M) \Big),$$

for all $m \geq M$, so $\int_{a,b)} g^2 dm = \int_a^b g^2 dm$ can only be ∞, and this clearly shows that g^2 can not be integrable.
c.) If $g = h$ a.e., we also have $\int_{(a,b)} |h|^2 dm = \infty$ for any interval (a, b). Thus, h can not be bounded on (a, b). Moreover, h is discontinuous at any point x, since otherwise h would be bounded on some interval containing x.

PROBLEM 53. Compute the following limits and justify the calculations:
a.) $\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} sin(\frac{x}{n})dx$;
b.) $\lim_{n\to\infty} \int_0^1 \left(1 + nx^2\right)\left(1 + x^2\right)^{-n} dx$;
c.) $\lim_{n\to\infty} \int_0^\infty nsin(\frac{x}{n})\left[x(1 + x^2)\right]^{-1} dx$;
d.) $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1} dx$, $(a \in R)$

SOLUTION.
a.) For $n \geq 2$ we have

$$\left|\left(1 + \frac{x}{n}\right)^{-n} sin(\frac{x}{n})\right| \leq \left(1 + \frac{x}{2}\right)^{-2}.$$

Note also that the function $\left(1 + \frac{x}{2}\right)^{-2}$ is integrable over $[0, \infty)$. Thus, by the Dominated Convergence Theorem ,we have

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} sin(\frac{x}{n})dx = \int_0^\infty \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{-n} sin(\frac{x}{n})dx = 0.$$

b.) $\left|\left(1 + nx^2\right)\left(1 + x^2\right)^{-n}\right| \leq 1$, and $\int_0^1 1dx = 1$. Thus by the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_0^1 \left(1 + nx^2\right)\left(1 + x^2\right)^{-n} dx = \int_0^1 \lim_{n\to\infty} \left(1 + nx^2\right)\left(1 + x^2\right)^{-n} dx = 0.$$

c.) $\left|nsin(\frac{x}{n})\left[x(1 + x^2)\right]^{-1}\right| \leq (1 + x^2)^{-1}$ and note that $(1 + x^2)^{-1}$ is integrable over $[0, \infty)$.Hence, again by the Dominated Convergence Theorem, we have,

$$\lim_{n\to\infty} \int_0^\infty nsin(\frac{x}{n})\left[x(1+x^2)\right]^{-1} dx = \int_0^\infty \lim_{n\to\infty} nsin(\frac{x}{n})\left[x(1+x^2)\right]^{-1} dx = \int_0^\infty (1+x^2)^{-1} = \frac{\pi}{2}.$$

d.) $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1} dx = \lim_{n\to\infty} \int_{na}^\infty (1 + y^2)^{-1} dy = \lim_{n\to\infty} \arctan(y)|_{na}^\infty = 0$ if $a > 0; = \frac{\pi}{2}$ if $a = 0; = \pi$ if $a < 0$.

PROBLEM 54. Suppose f_n and f are measurable complex-valued functions and $\phi : C \to C$.
a.) If ϕ is continuous and $f_n \to f$ a.e., then show that $\phi \circ f_n \to \phi \circ f$ a.e.
b.) Show that if ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$, uniformly, almost uniformly, or in measure, respectively.
c.) Give counterexamples when the continuity assumptions on ϕ are not satisfied.

SOLUTION.
a.) Since ϕ is continuous by assumption, $f_n \to f$ implies that $\phi \circ f_n \to \phi \circ f$, and

so $\{x : \phi \circ f_n \nrightarrow \phi \circ f\} \subset \{x : f_n \nrightarrow f\}$. Thus, $\mu(\{x : \phi \circ f_n \nrightarrow \phi \circ f\}) = 0$ as $\mu(\{x : f_n \nrightarrow f\}) = 0$.Therefore, $\phi \circ f_n \to \phi \circ f$ a.e.
b.) Since ϕ is uniformly continuous, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $|x - y| < \delta(\epsilon)$ implies that $|\phi(x) - \phi(y)| < \epsilon$. Now, if $f_n \to f$ uniformly, $\forall \epsilon > 0$, there is $M \in N$ such that when $n \geq M$, for all $x \in X$, $|f_n(x) - f(x)| < \delta(\epsilon)$, and so that $|\phi \circ f_n(x) - \phi \circ f| < \epsilon$. But this shows that $\phi \circ f_n \to \phi \circ f$, uniformly.
If $f_n \to f$ almost uniformly, then for any $\epsilon_1, \epsilon_2 > 0$, there is a set $E \in F(\sigma - algebra)$ and a natural number $M \in N$, such that $\mu(E) < \epsilon_1$, and when $n_j \geq M$ for $x \in X - E, |f_n(x) - f(x)| < \delta(\epsilon_2)$, and so $|\phi \circ f_n - \phi \circ f| < \epsilon_2$. This clearly shows that $\phi \circ f_n \to \phi \circ f$ almost uniformly.
If $f_n \to f$ in measure, then $\forall \epsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| > \delta(\epsilon)\}) \to)$. Since

$$\{x : |\phi \circ f_n - \phi \circ f| > \epsilon\} \subset \{x : |f_n - f| > \delta(\epsilon)\},$$

we have $\mu(\{x : |\phi \circ f_n - \phi \circ f| > \epsilon\}) \to 0$. But this means that $\phi \circ f_n \to \phi \circ f$, in measure.
c.) A counterexample for a.) is $f_n(x) = \frac{1}{n}, f(x) = 0$, and $\phi = 1_{\{0\}}$.
A counterexample for b.) is $X = R$, $f_n(x) = x + \frac{1}{n}, f(x) = x$ and $\phi(x) = x^2$.

PROBLEM 55. Suppose $f_n \to f$ almost uniformly, then show that $f_n \to f$ a.e. and in measure.

SOLUTION. First let us recall what it means to *converge almost uniformly*: It means, for all $\epsilon_1, \epsilon_2 > 0$, there is a set E such that $\mu(E) < \epsilon_1$ and $x \in (X - E)$ implies $|f_n(x) - f(x)| < \epsilon_2$.
Since $f_n \to f$ almost uniformly, for any $n \in N$, there is $E_n \in M(\sigma$-algebra such that $\mu(E_n) < \frac{1}{n}$ and $f_n \to f$ on E_N^c. Let $E = \bigcap_1^\infty E_n$, then $\mu(E) = 0$ and $f_n \to f$ on $\bigcup_1^\infty E_n^c = E^c$. Thus, $f_n \to f$ a.e.
Since $f_n \to f$ almost uniformly, for every $\epsilon_1, \epsilon_2 > 0$, there is $E \in M$ and $n_1 \in N$ such that $\mu(E) < \epsilon_2$ and when $n > n_1, |f_n(x) - f(x)| < \epsilon_1$ for x is not in E, and so $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon_1\}) \leq \mu(E) < \epsilon_2$. Thus,

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon_1\}) \to 0.$$

Therefore, $f_n \to f$ in measure as ϵ_1 and ϵ_2 are arbitrary.

PROBLEM 56. Show that if $f : [a, b] \to C$ is Lebesgue measurable and $\epsilon > 0$, then there is a set $E \subset [a, b]$ such that $m(E^c) < \epsilon$ and $f|_E$ is continuous. Moreover, E may be taken to be compact.

SOLUTION. Since $\bigcap_{n=1}^\infty \{x : |f(x)| > n\} = \emptyset$, there is $M \in N$ such that $m(\{x : |f(x)| > M\}) < \frac{\epsilon}{2}$. Let $E_1 = \{x : |f(x)| \leq M\}$, and define $h(x) = 1_{E_1} f(x)$. Now $h \in L^1[a, b]$, so we can find a subsequence of a sequence of continuous functions (g_n) which tends to f a.e.Without loss of generality we assume that $g_n \to h$ a.e. Applying Egoroff's Theorem, we have $E_2 \in M$ such that $m(E_2^c) < \frac{\epsilon}{2}$ and $g_n \to h$ uniformly on E_2. Then, we have that h is continuous on E_2 and so f is continuous on $E_1 \cap E_2$ for f

differs from h only on E_1. Now, $m((E_1 \cap E_2)^c) = m(E_1^c \cup E_2^c) < \epsilon$, there is an open set O containing $E_1^c \cup E_2^c$ and $m(O) < \epsilon$. Let $E = O^c$, then E is compact, $m(E^c) < \epsilon$ and $E \subset E_! \cap E_2$, so f is continuous on E.

PROBLEM 57. A measure μ is called *semi-finite* if every set of infinite measure contains a subset of finite, nonzero measure. Show that every $\sigma-$finite measure is semi-finite.

SOLUTION. Since μ is $\sigma-$finite, we know that

$$X = \bigcup_{j=1}^{\infty} E_j, \mu(E_j) < \infty.$$

Without loss of generality we may assume that E_j are disjoint. Let A be an arbitrary set with $\mu(A) = \infty$. Then,

$$\mu(A) = \sum_{j=1}^{\infty} \mu(A \cap E_j).$$

Each $A \cap E_j$ has finite measure as it is a subset of E_j. Since the sum is ∞, at least some of the sets $A \cap E_j$ must have nonzero measure(actually, infinitely many). Pick any of them.

PROBLEM 58. Let μ be the counting measure on N. Prove that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

SOLUTION. Assume $f_n \to f$ in measure. This means that for any $\epsilon > 0$ $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$ as $n \to \infty$. Since μ only takes integer values, this is equivalent to : $\exists N$ so that $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) = 0$ for $n > N$. This says, $|f_n(x) - f(x)| \leq \epsilon$ for $n > N$ and for all x. But this says $f_n \to f$ uniformly. The converse is obvious.

PROBLEM 59. Prove that for $a > 0$,

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(ax) dx = \sqrt{\pi} e^{-\frac{a^2}{4}}.$$

SOLUTION. Define,

$$f_n(x) = e^{-x^2} \sum_{j=0}^{n} (-1)^j \frac{(ax)^{2j}}{(2j)!},$$

and

$$g(x) = e^{-x^2} \sum_{j=0}^{\infty} (-1)^j \frac{(ax)^{2j}}{(2j)!} = e^{-x^2} \cosh(ax) = e^{-x^2} \frac{e^{ax} + e^{-ax}}{2}.$$

Then, $g \in L^1$, and $|f_n(x)| \leq g(x)$. Thus, we can use Dominated Convergence Theorem and integrate the series for f term by term. A simple calculation yields,

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{(ax)^{2n}}{(2n)!} dx = a^{2n} \frac{\sqrt{\pi}}{4^n n!}.$$

Thus,

$$\int_{-\infty}^{\infty} e^{-x^2} cos(ax) dx = \sum_{n=0}^{\infty} (-1)^n a^{2n} \frac{\sqrt{\pi}}{4^n n!} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-a^2/4)^n}{n!} = \sqrt{\pi} e^{-a^2/4}.$$

PROBLEM 60. Let (q_n) be an enumeration of rationals in $[0,1]$. Define the function f on $[0,1]$ by,

$$f(x) = \sum_{n, so, q_n < x} 2^{-n}.$$

(a.) Where is this function continuous/discontinuous?

(b.) Is this function Riemann integrable?

(c.) Is this function Lebesgue integrable?

SOLUTION.

(a.) If x is rational, then $x = q_n$ for some n. and $f(x+) - f(x-) = 2^{-n}$, so f is not continuous at x.
Let x be irrational. *Claim:* f is continuous at x. To prove the claim fix $\epsilon > 0$. Choose N so large that

$$\sum_{n=N}^{\infty} 2^{-n} < \epsilon.$$

Now, choose $\delta > 0$ so small that the interval $(x - \delta, x + \delta)$ does not contain any of the q_n with $n < N$. Then for $y \in (x - \delta, x + \delta)$,

$$|f(x) - f(y)| < \sum_{n > N} 2^{-n} < \epsilon.$$

(b.) The answer is yes. There is a theorem saying, if a function is bounded on a bounded interval and it has at most countably many points of discontinuity, then it is Riemann integrable. Our function satisfies the conditions so it is Riemann integrable.

(c.) Yes. It is Lebesgue integrable. For this we can use a theorem which says if f is properly Riemann integrable, then it is Lebesgue integrable.
Or we can try to do it directly. The partial sums

$$f_N(x) = \sum_{n \leq N, q_n < x} 2^{-n}$$

are non-negative, increasing and bounded above. So either by the Monotone Convergence Theorem or by the Dominated Convergence Theorem f is Lebesgue integrable.

PROBLEM 61. If f and g are two continuous functions on a common open set in R^n that agree everywhere on the complement of a set of zero Lebesgue measure,then, show that in fact f and g agree everywhere.

SOLUTION. Let f and g be two continuous functions such that $f(x) = g(x)$, for all $x \in A^c$ and $m(A) = 0$. Consider any point $a \in A$. Consider also the open ball $B(a,r) = \{y : |y-a| < r\}$. Since $m(A) = 0$ and $m(B(a,r)) > 0$ it is not possible to have $B(a,r) \subset A$ for any $r > 0$. Therefore, for all $r > 0, B(a,r)$ contains points in A^c. Thus, there exists a sequence of points each lying in A^c and converging to a.i.e. There exists $(a_n)_{n=1}^{\infty}$ such that $a_n \in A^c$ for all n and $a_n \to a$. But, then $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} g(a_n)$ since $f(a_n) = g(a_n)$ for all n. Since f and g are continuous on A^c the above equality gives,

$$f(\lim_{n\to\infty} a_n) = g(\lim_{n\to\infty} a_n)$$
$$\Rightarrow f(a) = f(\lim_{n\to\infty} a_n) = g(\lim_{n\to\infty} a_n) = g(a).$$

Since, $a \in A$ is arbitrary we have $g(a) = f(a)$ for all $a \in A$. Since also $f(b) = g(b)$ for all $b \in A^c$ we have $f(x) = g(x)$ for any $x \in R^n$.