Characterization of function spaces and boundedness of bilinear pseudodifferential operators through Gabor frames

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Characterization of function spaces and boundedness of bilinear pseudodifferential operators through Gabor frames

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To my family.

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SUMMARY

A frame in a separable Hilbert space H is a sequence of vectors $\{f_n\}_{n\in I}$ which provides a basis-like expansion for any vector in H. However, this representation is usually not unique, since most useful frames are over-complete systems, and, hence, are not bases. Furthermore, frames with particular structures—wavelet frames, exponential frames, or Gabor frames—have proven very useful in numerous applications.

Gabor frames, also known as Weyl-Heisenberg frames, are generated by time-frequency shifts of a single function which is called the window function or the generator. Not only do Gabor frames characterize any square integrable function, but they also provide a precise characterization of a class of Banach spaces called modulation spaces.

One objective of this thesis is to extend the theory of Gabor frames to other Banach spaces which are not included in the class of the modulation spaces. In particular, we will prove that Gabor frames do characterize a class of Banach spaces called amalgam spaces, which include the Lebesgue spaces and play important roles in sampling theory. Moreover, we will study the behavior of various operators connected to the theory of Gabor frames on the amalgam spaces.

Another objective of this thesis is to formulate and prove sufficient conditions on a function to belong to a particular modulation space. Modulation spaces have a rather implicit definition, yet they are the natural setting for time-frequency analysis. Consequently it is important to give sufficient conditions for membership in them. We will prove that certain classical Banach spaces such as the Besov and Triebel-Lizorkin spaces are embedded in the modulation spaces. These embeddings provide us with sufficient conditions for membership in the modulation spaces.

Finally, we will use the theory of Gabor frames to formulate certain boundedness results for bilinear pseudodifferential operators with non-smooth symbols on products of modulation spaces. More precisely, we use the Gabor frame expansions of functions in the modulation spaces to convert the boundedness of these operators to the boundedness of an infinite matrix acting on sequence spaces associated to the modulation spaces. A particular modulation space known as the Feichtinger algebra turns out to be a class of non-smooth symbols that yield the boundedness of the bilinear pseudodifferential operators on products on modulation spaces. Additionally, we use the same decomposition techniques to study the boundedness of the (linear) Hilbert transform on the modulation spaces in the one dimensional case.

CHAPTER I

PRELIMINARIES

1.1 Introduction

In 1946 D. Gabor [31] proposed a decomposition of signals that displays simultaneously the local time and frequency content of the signal, as opposed to the classical Fourier transform which displays only the global frequency content for the entire signal. He used building blocks generated by time-frequency shifts of a Gaussian, i.e., building blocks of the form $g_{m,n} = e^{2\pi i m \cdot} g(\cdot - n)$ where $g(x) = e^{-\pi x^2}$, and sought an orthonormal basis for $L^2(\mathbb{R})$ made up of these elementary functions. However, the Balian-Low theorem [17, Chapter 2] shows that no orthonormal basis can be obtained in this fashion with any function g as "nice" as the Gaussian. However, by relaxing the orthonormal basis requirement, and seeking a representation that preserves the main features of the signal —such as its energy— and that allows stable reconstruction, Gabor's idea turns out to yield positive results. More precisely, one can obtain using Gabor's scheme some very good and useful substitutes for orthonormal bases: Gabor frames.

Frames were introduced by R. J. Duffin and A. C. Schaeffer [15] in 1952 while working on some problems in nonharmonic Fourier series, but they were used little until the dawn of the wavelet era. Formally, a frame in a separable Hilbert space H is a sequence $\{f_n\}_{n\in I}$ for which there exist constants $0 < A, B < \infty$ —called frame bounds— such that

$$A\,\|f\|_H^2 \leq \sum_{n\in I} |\langle f, f_n\rangle|^2 \leq B\|f\|_H^2 \qquad \forall f \in H.$$

It is remarkable that the above inequalities imply the existence of a (canonical) dual frame $\{\tilde{f}_n\}_{n\in I}$, such that the following reconstruction formula holds for every $f \in H$:

$$f = \sum_{n\in I} \langle f, \tilde{f}_n \rangle \, f_n.$$

In particular, any orthonormal basis for H is a frame. However, in general, a frame need not be a basis and, in fact, most useful frames are over-complete. The redundancy that frames carry is what makes them very useful in many applications.

Gabor frames and wavelet frames are examples of "easily constructible" frames, and have played important roles in applications as well as in pure mathematics over the last two decades [14, 46]. From the abstract frame theory of Duffin and Schaeffer and Gabor's original idea, the theory of Gabor frames has grown to become a field on its own right. Notwithstanding the fact that many questions concerning the existence of Gabor frames remain unsolved, there exist numerous "constructible" examples of such frames whose generators are well-localized in the time-frequency plane. In such cases the frames are necessarily over-complete.

A more remarkable property of Gabor frames lies in the characterizations they provide for a whole class of Banach function spaces. Indeed, a deep result in the area, due to H. G. Feichtinger and K. Gröchenig [23, 24], is the atomic decomposition of the class of Banach spaces known as the modulation spaces via Gabor frames. These spaces were introduced by Feichtinger [21] and can be seen as the proper tools to quantify the time-frequency content of functions. The modulation spaces have since then found numerous applications. In particular, they appear quite naturally in the theory of pseudodifferential operators. A pseudodifferential operator is a formalism that assigns to a distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ —the symbol of the operator— a linear operator $T_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ in such a way that properties of the symbol can be inferred from properties of the operator. Moreover, pseudodifferential operators are encountered in engineering, where they are known as time-varying filters, as well as in quantum mechanics, where they appear under the name of quantization rules. We refer to cf. [41, Sect. 14.1] and the references therein for more background on pseudodifferential operators, as well as for their connections with partial differential equations. A natural question one could ask is to find conditions on σ under which T_σ can be extended to a bounded operator on L^2, or on more general Banach spaces. Symbols in the so-called Hörmander class are known to yield bounded pseudodifferential operators on various Banach spaces, cf. [26, Chapter 2]. In particular, Calderón-Vaillancourt [10] proved that if σ is smooth enough and has enough decay, then T_σ can be extended to a bounded operator on L^2. Gröchenig and Heil [42] recovered and extended this result using non-smooth symbols with only a mild time-frequency concentration as measured by a modulation space norm.

In this thesis we consider three problems centered around the theory of Gabor frames and the modulation spaces.

The first problem we consider is an extension of the theory of Gabor frames from its "natural setting" (the modulation spaces) to other spaces. Indeed, because the L^p spaces are not modulation spaces if $p \neq 2$ [25], it was not known if these spaces could be characterized via Gabor frames. In a joint work with K. Gröchenig and C. Heil [39], we show that Gabor frames do characterize a class of Banach spaces called the amalgam spaces, which include the Lebesgue spaces. Amalgam spaces are spaces that amalgamate local and global criterion for membership. They appear naturally in sampling theory, where they are the "right" setting for different problems [1]. Additionally, we will prove a weak necessary condition on the Gabor frame's generator, thereby extending a result due to R. Balan [2].

The second problem we consider is concerned with the modulation spaces. In spite of being the "right" spaces for time-frequency analysis, their rather implicit definition makes it very difficult to decide if a function belongs to a particular modulation space. We formulate sufficient conditions for membership in the modulation spaces by proving embeddings of certain Banach spaces such as the Besov and Triebel-Lizorkin spaces into some modulation spaces [50]. The class of Besov and Triebel-Lirzorkin spaces, includes some well-known Banach spaces such as the Lebesgue, the Hölder-Lipschitz, the Sobolev spaces, and is equipped with a wide variety of equivalent norms. We refer to [55, Chapter 4], [58, Chapters 1–2] for background on these spaces. The embedding results we prove can be seen as a comparison among certain properties of functions, i.e., smoothness and decay versus time-frequency concentration.

The last problem we consider can be viewed as an application of the theory of Gabor frames. More precisely, we consider the boundedness of bilinear pseudodifferential operators on modulation spaces. One of the motivation of investigating the bilinear pseudodifferential operators on the modulation spaces comes from the recent developments of their linear counterpart in the realm of the modulation spaces. More precisely, because the Weyl correspondence —which is a particular way of assigning to a symbol a pseudodifferential operator corresponding to the Weyl quantization rule— can be expressed as superposition of time-frequency shifts, which are the main objects used in defining the modulation spaces, it was natural to study the linear pseudodifferential operators on these spaces. Bilinear pseudodifferential operators are defined through their symbols as bilinear operators from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, and are not just generalizations of their linear counterparts, but are important tools in many problems in analysis [49]. A natural question in this context is again to find sufficient conditions on the symbols that guarantee the boundedness of the corresponding operators on products of certain Banach spaces. Smoothness and decay of the symbols are often the conditions needed to prove the boundedness of these operators [11, 49, 35, 36]. In a joint work with A. Bényi [6], we prove that if the symbols are in a particular modulation space —the so-called Feichtinger algebra— then the corresponding bilinear pseudodifferential operators are bounded on products of modulation spaces. As particular cases, we obtain boundedness results on products of certain Lebesgue spaces using non-smooth symbols. Finally, we prove that the Hilbert transform is bounded on the modulation spaces, using a discrete tool via the atomic decomposition of these spaces by Gabor frames and by relying on the L^2 theory of the Hilbert transform.

1.2 Outline of the Thesis

The thesis is organized as follows. Chapter 1 contains a brief survey of the notations and definitions that will be used in the sequel.

Chapter 2 is mostly expository. In particular, it contains the definition of the basic tools of time-frequency analysis. Moreover, it contains the definition of the Gabor frames, as well as their main properties. Additionally, the definition of the modulation spaces and their atomic decompositions by Gabor frames as well as their main properties is given in the chapter.

Chapter 3 is devoted to the first main result of the thesis, namely the characterization of the weighted amalgam spaces by Gabor frames. To obtain this characterization we study the behavior of the various operators connected with (Gabor) frames theory. Additionally we prove a weak necessary condition on the generator of the Gabor frames.

Chapter 4 contains the embedding results of certain Besov and Triebel-Lizorkin spaces into the modulation spaces. These results provide some sufficient conditions for membership in the modulation spaces. To obtain these results, we rely on the numerous equivalent norms defining the Besov and Triebel-Lizorkin spaces as well as the properties of the short time Fourier transform (STFT).

Finally, Chapter 5 contains some applications of the theory of Gabor frames to the study of bilinear pseudodifferential operators. In particular, we present in Section 5.1 a boundedness result for bilinear pseudodifferential operators using a discrete approach via Gabor frames. Section 5.2 is devoted to prove the boundedness of the Hilbert transform on the modulation spaces again using a discrete approach.

1.3 Notations

The usual dot product of $x, y \in \mathbb{R}^d$ is denoted by $x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n$. The length of x is $|x|$.

We use the notation $|a|$ to denote the magnitude of a complex number a.

We use the notation $f^*(t) = \overline{f(-t)}$.

The convolution of f and g is defined formally as $(f * g)(x) = \int f(x-t)\, g(t)\, dt$.

The Fourier transform of a function f is defined formally by

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)\, e^{-2\pi i t\cdot\omega}\, dt \qquad \text{for} \qquad \omega \in \mathbb{R}^d.$$

Similarly, the inverse Fourier transform of f is defined formally by

$$\mathcal{F}^{-1}f(t) = \check{f}(t) = \int_{\mathbb{R}^d} f(\omega)\, e^{2\pi i \omega\cdot t}\, d\omega \qquad \text{for} \qquad t \in \mathbb{R}^d.$$

If $E \subset \mathbb{R}^d$ is a measurable set, χ_E is the characteristic function of E, and we denote its Lebesgue measure by $|E|$.

If $a > 0$, we deonote Q_a the cube in $\mathbb{R}^d$ with side length a, i.e., $Q_a = [0, a[^d$.

Let X be a Banach space, then the norm of $u \in X$ will be denoted $\|u\|_X$ or simply $\|u\|$ when the appropriate space is clear from the context. Moreover, if two norms $\|\cdot\|^1$ and $\|\cdot\|^2$, are equivalent on a Banach space X we will write $\|u\|^1 \asymp \|u\|^2$ to mean the existence of two positive constant C_1, C_2 such that

$$C_1\|u\|^1 \le \|u\|^2 \le C_2\|u\|^1 \qquad \forall u \in X.$$

The dual of a Banach space X is denoted X^*. We write $\langle f, g\rangle$ for the action of $f \in X'$ on $g \in X$.

The adjoint of an operator T is denoted by T^*.

For $1 \le p \le \infty$, p' will denote the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

$L^p(\mathbb{R}^d)$ is the Banach space of complex-valued functions f on $\mathbb{R}^d$ with norm

$$\|f\|_p = \|f\|_{L^p} = \left(\int_{\mathbb{R}^d} |f(x)|^p\, dx\right)^{1/p},$$

for $1 \leq p < \infty$. If $p = \infty$, the norm is given by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|.$$

Similarly, $\ell^p(\mathbb{Z}^d)$ is the Banach space of complex-valued sequences c on $\mathbb{Z}^d$ with norm

$$\|c\|_p = \|c\|_{\ell^p} = \left(\sum_{n \in \mathbb{Z}^d} |c_n|^p \right)^{1/p},$$

for $1 \leq p < \infty$. If $p = \infty$, the norm is given by

$$\|c\|_\infty = \sup_{n \in \mathbb{Z}^d} |c_n|.$$

We will also consider weighted mixed-norm spaces $L^{p,q}_\nu(\mathbb{R}^{2d})$, which are Banach spaces of complex-valued functions f on $\mathbb{R}^{2d}$ with norm

$$\|f\|_{L^{p,q}_\nu} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,y)|^p \, \nu(x,y)^p \, dx \right)^{q/p} dy \right)^{1/q},$$

for $1 \leq p, q < \infty$, with obvious modifications if $p = \infty$, or $q = \infty$. The weight function ν will be described in the following chapters.

We define similarly the discrete weighted mixed-norm spaces $\ell^{p,q}_{\tilde{\nu}}$ as the Banach space of complex-valued sequences c on $\mathbb{Z}^{2d}$ with norm

$$\|c\|_{\ell^{p,q}_\nu} = \left(\sum_{l \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |c_{k,l}|^p \, \tilde{\nu}(k,l)^p \right)^{q/p} \right)^{1/q},$$

for $1 \leq p, q < \infty$, with usual modifications if $p = \infty$ or $q = \infty$. The weight $\tilde{\nu}$ is an appropriate sample of the weight function ν, typically $\tilde{\nu}(k,l) = \nu(\alpha k, \beta l)$ for some $\alpha, \beta > 0$.

We use the notation ω_s for the function $\omega_s(x) = (1 + |x|^2)^{s/2}$ for $s > 0$.

If E is a measurable subset of $\mathbb{R}^d$, we let

$$\|f\|_{p,E} = \|f \, \chi_E\|_p$$

denote the norm of the function f restricted to the set E.

If T is a bounded linear operator from a Banach space X to a Banach space Y, we denote the operator norm of T by $\|T\|_{X \to Y}$, or simply by $\|T\|$ if there is no confusion.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, we write $|\alpha| = \sum_{i=1}^d \alpha_i$. The differentiation operator D^α and the multiplication operator X^β are defined by

$$D^\alpha f(x) = \prod_{i=1}^d (\partial_{x_i})^{\alpha_i} f(x), \qquad \text{and} \qquad X^\beta f(x) = \prod_{i=1}^d x_i^{\beta_i} f(x).$$

$\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of all infinitely differentiable functions f for which the seminorms

$$\|f\|_{(M,N)} = \sum_{|\alpha| \leq M} \sum_{|\beta| \leq N} \|D^\alpha X^\beta f\|_\infty$$

are finite for all non-negative integers M, N. Its topological dual, $\mathcal{S}'(\mathbb{R}^d)$, is the space of tempered distributions.

More details on the basic properties of the Fourier transform and more generally, on some of the theory from real and functional analysis that we will systematically used in the sequel can be found in many standard analysis texts, e.g., [27], [48], [52].

CHAPTER II

GABOR FRAMES AND MODULATION SPACES

A Gabor frame $\mathcal{G}(g, \alpha, \beta) = \{e^{2\pi i \beta n \cdot} g(\cdot - \alpha k)\}_{k,n \in \mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$ provides basis-like series representations of functions in L^2, with unconditional convergence of the series. However, unless the frame is a Riesz basis (and hence, by the Balian–Low theorem has poor time-frequency localization), these representations will not be unique. Still, a canonical and computable representation exists, and Gabor frames have found a wide variety of applications in mathematics, science, and engineering [14, 17, 18, 41]. An important fact is that Gabor frames provide much more than just a means to recognize square-integrability of functions. If the window function g is reasonably well-localized in time and frequency, then Gabor frame expansions are valid not only in L^2 but in an entire range of associated spaces $M^{p,q}_\nu$ known as the modulation spaces. The frame expansions converge unconditionally in the norm of those spaces, and membership of a tempered distribution in $M^{p,q}_\nu$ is *characterized* by membership of its sequence of Gabor coefficients in a weighted sequence space $\ell^{p,q}_{\tilde{\nu}}$. We refer to [41] for a recent detailed development of time-frequency analysis and modulation spaces.

In this chapter, we review some of the key results regarding Gabor expansions of L^2 functions. We then define the modulation spaces and their atomic decompositions by Gabor frames, which will be used often throughout this thesis.

2.1 Weight Functions

Before delving into Gabor analysis per se, we introduce here a class of weight functions that appear in most of the subsequent chapters.

2.1.1 Submultiplicative weights

A *submultiplicative weight function* ω is a positive, symmetric, and continuous function which satisfies

$$\forall\, x, y \in \mathbb{R}^d, \quad \omega(x+y) \;\le\; \omega(x)\, \omega(y).$$

The prototypical example of a submultiplicative weight is the polynomially-growing function $\omega_s(x) = (1+|x|^2)^{s/2}$, where $s > 0$. We also consider weight functions defined on $\mathbb{R}^{2d}$ by making the obvious changes in the definition.

2.1.2 Moderate weights

A positive, symmetric, and continuous function ν is called *ω-moderate function* if there exists a constant $C_\nu > 0$ such that

$$\forall\, x, y \in \mathbb{R}^d, \quad \nu(x+y) \;\le\; C_\nu\, \omega(x)\, \nu(y). \tag{1}$$

If ν is ω-moderate with respect to $\omega = \omega_s$, for some $s > 0$, we say that ν is s-moderate.

For example, $\nu(x) = (1+|x|)^t$ is moderate with respect to $\omega_s(x) = (1+|x|^2)^{s/2}$, where $s > 0$, exactly for $|t| \le s$.

If desired, the assumptions of continuity and symmetry of ω and ν could be removed, but there would be no increase in the generality of the results. For if ω is a positive, submultiplicative function, then there exists a continuous weight function ω_1 such that $0 < A \le \omega(x)/\omega_1(x) \le B < \infty$ for all x, and similarly for ω-moderate functions ν, cf. [41, Sect. 11.1].

If ν is ω-moderate, then by manipulating (1) we see that

$$\frac{1}{\nu(x+y)} \le C_\nu\, \omega(x)\, \frac{1}{\nu(y)},$$

so $1/\nu$ is also ω-moderate (with the same constant). Thus, the class of ω-moderate weights is closed under reciprocals, and consequently the class of spaces L^p_ν using ω-moderate weights is closed under duality (with the usual exception for $p = \infty$). This would not be the case if we restricted only to submultiplicative weights.

Given an ω-moderate weight ν on $\mathbb{R}^d$, we will often use the notation $\tilde{\nu}$ to denote the weight on $\mathbb{Z}^d$ defined by $\tilde{\nu}(k) = \nu(\alpha k)$, and for a weight ν on $\mathbb{R}^{2d}$ we define $\tilde{\nu}(k,n) = \nu(\alpha k, \beta n)$, or $\tilde{\nu}(k) = \nu(k/\beta)$, the particular choice being clear from context,

2.2 Gabor frames in L^2

Before defining Gabor frames we first introduce two operators that play important roles in time-frequency analysis.

Definition 2.2.1. Given $a, b \in \mathbb{R}^d$, the translation and modulation operators are defined respectively by

$$T_a f(t) = f(t-a), \qquad M_b f(t) = e^{2\pi i t\cdot b} f(t)$$

for any function f defined on $\mathbb{R}^d$.

Additionally, for $c \in \mathbb{R}, c \neq 0$ we define the dilation operator acting on a function f defined on $\mathbb{R}^d$ by

$$D_c f(t) = |c|^{-d/2} f(t/c).$$

It is easily seen that the translation, modulation, and dilation operators are unitary on L^2, and that they map $\mathcal{S}$ and $\mathcal{S}'$ isomorphically onto themselves. The following lemma collects some basic facts about the translation and modulation operators.

The proof of the following lemma is immediate from the definition so we omit it.

Lemma 2.2.2. *Let $a, b \in \mathbb{R}^d$, $c \in \mathbb{R}, c \neq 0$, and f be a function defined on $\mathbb{R}^d$. The following statements hold.*

a. $T_a M_b f = e^{-2\pi i a\cdot b} M_b T_a f$.

b. $D_c T_a f = T_{ca} D_c f$.

c. $D_c M_b f = M_{\frac{b}{c}} D_c f$.

d. $\widehat{T_a f} = M_{-a} \hat{f}$, *and* $\widehat{M_b f} = T_b \hat{f}$.

e. $\widehat{D_c f} = D_{\frac{1}{c}} \hat{f}$.

f. $\widehat{M_b T_a} f = e^{2\pi i a \cdot b} M_{-a} T_b \hat{f}$.

We are now in position to define Gabor frames.

Definition 2.2.3. Given a *window function* $g \in L^2(\mathbb{R}^d)$ and given α, $\beta > 0$, we say that

$$\mathcal{G}(g, \alpha, \beta) \;=\; \{M_{\beta n} T_{\alpha k} g\}_{k,n \in \mathbb{Z}^d} \;=\; \{e^{2\pi i \beta n \cdot} g(\cdot - \alpha k)\}_{k,n \in \mathbb{Z}^d}$$

is a *Gabor frame* for $L^2(\mathbb{R}^d)$ if there exist constants A, $B > 0$ (called *frame bounds*) such that for all $f \in L^2(\mathbb{R}^d)$,

$$A \, \|f\|_{L^2}^2 \;\leq\; \sum_{k,n \in \mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha k} g \rangle|^2 \;\leq\; B \, \|f\|_{L^2}^2. \tag{2}$$

We should point out that due to the commutation relations between the translation and modulation operators, it is trivial to see that the order of the translation and modulation in the definition of a Gabor frame is irrelevant. Moreover, the image of a Gabor frame $\mathcal{G}(g, \alpha, \beta)$ under the Fourier transform is another Gabor frame, namely $\mathcal{G}(\hat{g}, \beta, \alpha)$.

Definition 2.2.4. Consider the collection of time-frequency shifts $\mathcal{G}(g, \alpha, \beta)$ generated by $g \in L^2(\mathbb{R}^d)$, and $\alpha, \beta > 0$.

a. The analysis operator associated with $\mathcal{G}(g, \alpha, \beta)$ is the operator $C_g : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^{2d})$ defined by $C_g f = \big(\langle f, M_{\beta n} T_{\alpha k} g \rangle\big)_{k,n \in \mathbb{Z}^d}$, for $f \in L^2$.

b. The synthesis operator associated with $\mathcal{G}(g, \alpha, \beta)$ is the operator $R_g : \ell^2(\mathbb{Z}^{2d}) \to L^2(\mathbb{R}^d)$ defined formally by $R_g c = \sum_{k,n \in \mathbb{Z}^d} c_{kn} \, M_{\beta n} T_{\alpha k} g$, for $c = (c_{k,n})_{k,n \in \mathbb{Z}^d} \in \ell^2$.

The basic properties of Gabor frames are laid out in the following result; we refer to [14], [41], or [46] for more extensive treatments of frames and Gabor frames.

Theorem 2.2.5. *Let $\mathcal{G}(g, \alpha, \beta)$ be a Gabor frame for $L^2(\mathbb{R}^d)$ with frame bounds A, B. Then the following statements hold.*

a. *The* analysis operator $C_g f = \big(\langle f, M_{\beta n} T_{\alpha k} g\rangle\big)_{k,n\in\mathbb{Z}^d}$ *is a bounded mapping from L^2 into ℓ^2, and we have the norm equivalence $\|f\|_2 \asymp \|C_g f\|_{\ell^2}$.*

b. *The* synthesis operator $R_g c = \sum_{k,n\in\mathbb{Z}^d} c_{kn}\, M_{\beta n} T_{\alpha k} g$ *is a bounded mapping from ℓ^2 into L^2. The series defining $R_g c$ converges unconditionally in L^2 for every $c \in \ell^2$.*

c. $R_g = C_g^*$*, and the* frame operator $S_g = R_g C_g \colon L^2 \to L^2$ *is strictly positive.*

d. *The* dual window $\gamma = S_g^{-1} g$ *generates a Gabor frame $\mathcal{G}(\gamma, \alpha, \beta)$ for $L^2(\mathbb{R}^d)$ with frame bounds $1/B$, $1/A$.*

e. $R_\gamma C_g = I$ *on $L^2(\mathbb{R}^d)$, i.e., we have the* Gabor expansions

$$f \;=\; R_\gamma C_g f \;=\; \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} \gamma\rangle\, M_{\beta n} T_{\alpha k} g \tag{3}$$

for $f \in L^2(\mathbb{R}^d)$, with unconditional convergence of the series.

Proof. a. The fact that $C_g : L^2 \to \ell^2$ is bounded follows from the second part of (2); moreover, (2) is precisely the statement that $\|f\|_2 \asymp \|C_g f\|_{\ell^2}$.

b. Let $c = (c_{kn})_{k,n\in F}$, where F is a finite subset of $\mathbb{Z}^{2d}$. For $f \in L^2$, we have

$$\begin{aligned}
|\langle R_g c, f\rangle| &= |\sum_{k,n\in F} c_{kn} \langle M_{\beta n} T_{\alpha k} g, f\rangle| \\
&\le \sum_{k,n\in F} |c_{kn}| |\langle f, M_{\beta n} T_{\alpha k} g\rangle| \\
&\le \Big(\sum_{kn\in F} |c_{kn}|^2\Big)^{1/2} \Big(\sum_{kn\in \mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha k} g\rangle|^2\Big)^{1/2} \\
&\le \sqrt{B}\, \|f\|_{L^2} \Big(\sum_{kn\in F} |c_{kn}|^2\Big)^{1/2},
\end{aligned}$$

where we have used the second inequality in (2). By duality we obtain

$$\begin{aligned}\|R_g c\|_{L^2} &= \sup_{\|f\|_{L^2}=1} |\langle R_g c, f\rangle| \\ &\le \sqrt{B}\left(\sum_{kn\in F} |c_{kn}|^2\right)^{1/2}\end{aligned}$$

for all sequences with finite support. A standard density argument shows that R_g is bounded from ℓ^2 to L^2, and that the series defining R_g converges unconditionally.

c. For $f \in L^2$ and $c = (c_{kn})_{k,n\in\mathbb{Z}^d}$, we have

$$\langle R_g c, f\rangle = \sum_{k,n\in\mathbb{Z}^d} c_{kn}\overline{\langle f, M_{\beta n}T_{\alpha k}g\rangle}\langle c, C_g f\rangle.$$

Hence, $R_g = C_g^*$. The frame operator $S_g = R_g C_g$ is clearly bounded on L^2; moreover, the first part of inequality (2) implies that S_g is strictly positive.

d. The frame inequality (2) can be rewritten as $A\,\|f\|_{L^2}^2 \le \langle S_g f, f\rangle \le B\,\|f\|_{L^2}^2$ for all $f \in L^2$, or equivalently in operator notation as $A\,I \le S_g \le B\,I$. Moreover, the above operator inequalities are preserved when multiplied by operators that commute with each of the terms appearing in the inequalities. Thus, we obtain that $B^{-1}\,I \le S_g^{-1} \le A^{-1}\,I$. Moreover, by some easy computations, one can show that S_g commutes with the translation and modulation operators $T_{\alpha k}$ and $M_{\beta n}$, and so does S_g^{-1}. Hence, $S_g^{-1} = S_\gamma$, which together with the last operator inequality concludes the proof of this part.

e. Follows from the fact that $f = S_g S_\gamma f$, and that S_g and S_γ commute with the translation and modulation operators $T_{\alpha k}$ and $M_{\beta n}$.

□

In brief, if $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$ then the ℓ^2-norm of the sequence of Gabor coefficients $(\langle f, M_{\beta n}T_{\alpha k}g\rangle)_{k,n\in\mathbb{Z}^d}$ is an equivalent norm for L^2, and the Gabor expansions given by (3) hold in L^2. Moreover, for our purposes it is important to note that once the analysis and synthesis operators are defined, the statement "Gabor expansions converge in L^2" is equivalent to the statement that the identity operator on L^2 factorizes as $I = R_\gamma C_g$.

In all these statements, and throughout this thesis, the roles of g and γ may be interchanged.

2.3 Modulation spaces

2.3.1 Definition and basic properties

The modulation spaces introduced by Feichtinger are spaces of tempered distributions defined by imposing some decay condition on their short-time Fourier transforms, which we next define.

Definition 2.3.1. The Short-Time Fourier Transform (STFT) of a function $f \in L^2$ with respect to a window $g \in L^2$ is

$$V_g f(x,y) \;=\; \langle f, M_y T_x g\rangle \;=\; \int_{\mathbb{R}^d} e^{-2\pi i y\cdot t}\, \overline{g(t-x)}\, f(t)\, dt.$$

Remark 2.3.2. From the above definition, it is clear that the STFT can be defined whenever f and g are in dual spaces. In particular, the STFT is well-defined and scalar-valued when $f \in \mathcal{S}'$ and $g \in \mathcal{S}$. Moreover, analogously to the Fourier transform, the STFT extends in a distributional sense to f, g in the space of tempered distributions $\mathcal{S}'$, cf. [26, Prop. 1.42].

The next proposition, whose proof is immediate and will be omitted, collects some different definitions of the STFT that will be used throughout this thesis.

Proposition 2.3.3. *If $f, g \in L^2(\mathbb{R}^d)$, then the following statements are true:*

$$\begin{aligned}
V_g f(x, y) &= \langle f, M_y T_x g\rangle \\
&= \big(f \cdot T_x \overline{g}\big)^{\wedge}(y) \\
&= e^{2\pi i x\cdot y}\, f * \big(M_y g^*\big)(x) \\
&= \langle \hat{f}, T_y M_{-x} \hat{g}\rangle \\
&= e^{-2\pi i x\cdot y}\, \langle \hat{f}, M_{-x} T_y \hat{g}\rangle \\
&= e^{-2\pi i x\cdot y}\, V_{\hat{g}} \hat{f}(y, -x).
\end{aligned}$$

The next result sheds some light on the behavior of the STFT on L^2.

Proposition 2.3.4. *Let $g \in L^2(\mathbb{R}^d)$, and assume that $g \neq 0$. Then for all $f \in L^2(\mathbb{R}^d)$ we have that*

$$\|V_g f\|_{L^2} = \|f\|_{L^2}\, \|g\|_{L^2}.$$

That is, V_g is a multiple of an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$.

Proof. First assume that $f \in \mathcal{S}$ then, $f \cdot T_x \overline{g} \in L^2(\mathbb{R}^d)$ for almost all $x \in \mathbb{R}^d$. Therefore, we have

$$\begin{aligned}
\|V_g f\|_{L^2}^2 &= \iint_{\mathbb{R}^{2d}} |V_g f(x, \omega)|^2\, dx\, d\omega \\
&= \iint_{\mathbb{R}^{2d}} |(\widehat{f \cdot T_x \overline{g}})(\omega)|^2\, d\omega\, dx \\
&= \iint_{\mathbb{R}^{2d}} |(f \cdot T_x \overline{g})(t)|^2\, dt\, dx \\
&= \iint_{\mathbb{R}^{2d}} |f(t)|^2\, |g(t-x)|^2\, dt\, dx \\
&= \|f\|_{L^2}^2\, \|g\|_{L^2}^2.
\end{aligned}$$

Thus $\|V_g f\|_{L^2} = \|f\|_{L^2}\, \|g\|_{L^2}$ for all $f \in \mathcal{S}$, and a standard density argument extends the result to all $f \in L^2$. □

The previous proposition shows that on L^2 the STFT is an isometry (up to a constant), and hence does not provide any new information beside the conservation of the energy. However, by imposing other norms on the STFT, we can better quantify the time-frequency concentration of functions. More precisely, we have the following definition.

Definition 2.3.5. Let ν be an ω-moderate weight on $\mathbb{R}^{2d}$, and let $1 \leq p, q \leq \infty$. Given a window function $g \in \mathcal{S}$, the modulation space $M_\nu^{p,q}(\mathbb{R}^d)$ is the space of all distributions $f \in \mathcal{S}'$ for which the following norm is finite:

$$\|f\|_{M_\nu^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p \, \nu(x,\xi)^p dx\right)^{q/p} d\xi\right)^{1/q} = \|V_g f\|_{L_\nu^{p,q}}, \tag{4}$$

with the usual modifications when p or q is infinite.

For background and information on the basic properties of the modulation spaces we refer to [21], [23], [24], [41]. The definition of the modulation space is independent of the choice of the window g in the sense of equivalent norms. More precisely, the following result whose prove may be found in [41, Proposition 11.3.1].

Proposition 2.3.6. *Assume that ν is ω-moderate and that $g_1, g_2 \in \mathcal{S}(\mathbb{R}^d)$ and that $g_i \neq 0$ when $i = 1, 2$. If $1 \leq p, q \leq \infty$, let $\|f\|_{M_\nu^{p,q}}^{g_i}$ denote the norm of f in the modulation space $M_\nu^{p,q}$ as measured by the window g_i when $i = 1, 2$. Then there exist two constants $C_1, C_2 > 0$ such that*

$$C_1 \frac{1}{\|V_{g_2} g_1\|_{L^1_\omega}} \|f\|_{M_\nu^{p,q}}^{g_2} \leq \|f\|_{M_\nu^{p,q}}^{g_1} \leq C_2 \|V_{g_1} g_2\|_{L^1_\omega} \|f\|_{M_\nu^{p,q}}^{g_2}.$$

The next theorem collects some basic facts on the modulation spaces, its proof may be found in [41].

Theorem 2.3.7. *Let ν be an ω-moderate weight.*

a. *For $1 \leq p, q \leq \infty$, $M_\nu^{p,q}$ is a Banach space.*

b. *If $p, q < \infty$, $\mathcal{S}$ is a dense subspace of $M^{p,q}_{\nu}$. Moreover, the dual of $M^{p,q}_{\nu}$ is the modulation space $M^{p',q'}_{1/\nu}$. More precisely we have that*

$$\|f\|_{M^{p,q}_{\nu}} = \sup_{\|g\|_{M^{p',q'}_{1/\nu}}=1} |\langle f, g\rangle|.$$

c. *If $p_1 \leq p_2$, and $q_1 \leq q_2$, then $M^{p_1,q_1}_{\nu} \subset M^{p_2,q_2}_{\nu}$.*

d. *If $p, q < \infty$, then M^1_{ω} is a dense subspace of $M^{p,q}_{\nu}$.*

Remark 2.3.8. a. If $p = q$ we denote the modulation space $M^{p,p}_{\nu}$ simply by M^p_{ν}. Moreover, if $\nu = 1$ we simply denote $M^{p,q}_{\nu}$ by $M^{p,q}$.

b. Among the modulation spaces are certain well-known spaces:

- if $\nu(x,\xi) = 1$, and $p = q = 2$, it is easy to see that $M^2 = L^2$,
- if $\nu(x,\xi) = (1 + |x|^2)^{s/2}$ where $s > 0$, and $p = q = 2$, then $M^2_{\nu} = L^2_s$, a weighted-L^2 space,
- if $\nu(x,\xi) = (1 + |\xi|^2)^{s/2}$ where $s > 0$, and $p = q = 2$, then $M^2_{\nu} = H^2_s$, the standard Sobolev space.

c. L^p for $p \neq 2$ does not coincide with any modulation space [25].

d. The modulation M^1 has several properties that deserve to be mentioned. It is a Banach algebra under both pointwise multiplication and convolution. It is the smallest Banach space that is isometrically invariant under translation and modulation. Moreover, it is a Segal algebra known as the Feichtinger algebra, and often denoted S_0, and plays an important role in time-frequency analysis. We refer to [41] and the references therein for more detail on the Feichtinger algebra and its weighted version.

The next result, whose proof can be found in [41, Proposition 11.3.1], provides a characterization of $\mathcal{S}$ and its dual $\mathcal{S}'$ in terms of the modulation spaces.

Proposition 2.3.9. *Let v_s be the weight function defined on $\mathbb{R}^{2d}$ by $v_s(z) = (1 + |z|)^s, z \in \mathbb{R}^{2d}$. Then we have*

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{s\geq 0} M^\infty_{v_s} \qquad \text{and} \qquad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{s\geq 0} M^\infty_{1/v_s}.$$

The next proposition, due to Feichtinger [21], on complex interpolation of modulation spaces will be used in the proof of our results in the following chapters. For more background on complex interpolation we refer to [8, Chapter 4].

Proposition 2.3.10. *Let $1 \leq p_0 < \infty$, $1 \leq q_0 < \infty$, $1 \leq p_1 \leq \infty$, $1 \leq q_1 \leq \infty$, and $\theta \in (0,1)$. If $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ then*

$$\left(M^{p_0,q_0}, M^{p_1,q_1}\right)_{[\theta]} = M^{p,q}. \tag{5}$$

2.3.2 Gabor frames on modulation spaces

Under stronger assumptions on g, the expansions in (3) are valid not only in L^2 but in the entire class of the modulation spaces.

The following result summarizes some basic facts on Gabor frames in the modulation spaces, cf. [41, Ch. 12]. The theorem is not stated in its weakest possible form; for example, the boundedness of the analysis and synthesis operators requires only the assumption $g \in M^1_\omega$, and does not require that g generate a frame for L^2. Recall that the mixed-norm sequence space $\ell^{p,q}_{\tilde{\nu}}$ consists of all sequences $c = (c_{kn})_{k,n\in\mathbb{Z}^d}$ such that

$$\|c\|_{\ell^{p,q}_{\tilde{\nu}}} \;=\; \left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} |c_{kn}|^p\, \tilde{\nu}(k,n)^p\right)^{q/p}\right)^{1/q} \;<\; \infty,$$

where $\tilde{\nu}(k,n) = \nu(\alpha k, \beta n)$, with the usual adjustments when $p = \infty$ or $q = \infty$.

Theorem 2.3.11. *Let ν be an ω-moderate weight on $\mathbb{R}^{2d}$, and let $1 \leq p,q \leq \infty$. Let $g \in M^1_\omega$ be such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Then the following statements hold.*

a. *The analysis operator defined by* $C_g f = \big(\langle f, M_{\beta n} T_{\alpha k} g\rangle\big)_{k,n\in\mathbb{Z}^d}$ *is a bounded mapping from* $M^{p,q}_\nu$ *to* $\ell^{p,q}_{\tilde\nu}$*, and we have the norm equivalence*

$$\|f\|_{M^{p,q}_\nu} \asymp \|C_g f\|_{\ell^{p,q}_{\tilde\nu}}.$$

b. *The synthesis operator defined by* $R_g c = \sum_{k,n\in\mathbb{Z}^d} c_{kn}\, M_{\beta n} T_{\alpha k} g$ *is a bounded mapping from* $\ell^{p,q}_{\tilde\nu}$ *to* $M^{p,q}_\nu$*. The series defining* $R_g c$ *converges unconditionally in the norm of* $M^{p,q}_\nu$ *for every* $c \in \ell^{p,q}_{\tilde\nu}$ *(weak* unconditionally in* $M^{\infty,\infty}_{1/\omega}$ *if* $p = \infty$ *or* $q = \infty$*).*

c. *The frame operator* $S_g = R_g C_g$ *is a continuously invertible mapping of* $M^{p,q}_\nu$ *onto itself.*

d. *The dual window* $\gamma = S_g^{-1} g$ *lies in* M^1_ω*.*

e. $R_\gamma C_g = I$ *on* $M^{p,q}_\nu$*, i.e., we have the* Gabor expansions

$$f \;=\; R_\gamma C_g f \;=\; \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k}\gamma\rangle\, M_{\beta n} T_{\alpha k} g \tag{6}$$

for $f \in M^{p,q}_\nu$*, with unconditional convergence of the series if* $p, q < \infty$*, and with unconditional weak* convergence otherwise.*

f. *A distribution* $f \in M^{\infty,\infty}_\nu$ *belongs to* $M^{p,q}_\nu$ *if and only if* $C_g f \in \ell^{p,q}_{\tilde\nu}$*. If* $g \in \mathcal{S}$*, then a tempered distribution* $f \in \mathcal{S}'(\mathbb{R}^d)$ *belongs to* $M^{p,q}_\nu$ *if and only if* $C_g f \in \ell^{p,q}_{\tilde\nu}$*.*

In brief, the $\ell^{p,q}_{\tilde\nu}$ norm of the Gabor coefficients $(\langle f, M_{\beta n} T_{\alpha k} g\rangle)_{k,n\in\mathbb{Z}^d}$ is an equivalent norm for $M^{p,q}_\nu$, and the Gabor expansions (3) are valid in $M^{p,q}_\nu$ with unconditional convergence of that series in the norm of $M^{p,q}_\nu$. Moreover, there is a strong statement made in part f of Theorem 2.3.11 that is not usually observed in the standard list of Gabor frame properties in L^2 (Theorem 2.2.5), namely that $\|C_g f\|_{\ell^{p,q}_{\tilde\nu}}$ is not only an equivalent norm for $M^{p,q}_\nu$, but membership of f in the modulation space is *characterized* by membership of its sequence of Gabor coefficients $C_g f$ in $\ell^{p,q}_{\tilde\nu}$. In particular, only the magnitude of these coefficients is important in determining whether a given distribution lies in $M_\nu^{p,q}$.

The proof of Theorem 2.3.11 requires deep analysis. In particular, the invertibility of S_g on M_ω^1 for arbitrary values of α, β was only recently proved in [43].

In summary, once the analysis and synthesis operators have been correctly defined, the fact that Gabor expansions converge in the modulation spaces is simply the statement that the identity operator on $M_\nu^{p,q}$ factorizes as $I = R_\gamma C_g$.

CHAPTER III

GABOR ANALYSIS IN WEIGHTED AMALGAM SPACES

Some results on Gabor analysis outside of the modulation spaces were obtained by Walnut in [59]. In particular, he introduced what is now known as the *Walnut representation* of the frame operator, and considered the boundedness of the frame operator on L^p. Recently, it was independently observed in [34] and [38] that Gabor expansions actually converge in $L^p(\mathbb{R}^d)$ when $1 < p < \infty$. Since L^p is not a modulation space when $p \neq 2$, it was known that Gabor expansions could not converge unconditionally in L^p [25].

In this chapter we consider a much larger class of spaces than the L^p spaces, namely, we consider the weighted *amalgam spaces* $W(L^p, L^q_\nu)$. These spaces amalgamate a local criteria for membership with a global criteria. We will show that not only do Gabor expansions converge for the special case $L^p = W(L^p, L^p)$, but that they converge in the entire range of weighted amalgam spaces. Moreover, membership in the amalgam space is characterized by membership of the Gabor coefficients in an appropriate sequence space. In the course of obtaining these results, we prove several results of independent interest on the behavior of the analysis and synthesis operators associated with the Gabor frame, and on the Walnut representation, which is an extremely useful tool in Gabor frame theory. Moreover, we include the cases $p = 1, \infty$ or $q = 1, \infty$ in our consideration. In particular, we show that Gabor expansions exist even in L^1 and in a weak sense in L^∞, given the right interpretation of "expansion." Additionally, we obtain some necessary conditions on the window

g, extending weaker necessary conditions obtained by Balan in [2] for the particular case $W(L^2, L^\infty)$.

3.1 Weighted amalgam spaces

3.1.1 Definition

Given an ω-moderate weight ν on $\mathbb{R}^d$ and given $1 \le p, q \le \infty$, the weighted amalgam space $W(L^p, L^q_\nu)$ is the Banach space of all measurable functions on $\mathbb{R}^d$ for which the norm

$$\|f\|_{W(L^p,L^q_\nu)} = \left(\sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_p^q \, \nu(\alpha k)^q\right)^{1/q} \tag{7}$$

is finite, with the usual adjustment if $q = \infty$.

The first use of amalgam spaces was by Wiener, who introduced the spaces $W(L^1, L^2)$ and $W(L^2, L^1)$ in [60] and $W(L^\infty, L^1)$ and $W(L^1, L^\infty)$ in [61], [62], in connection with his development of the theory of generalized harmonic analysis. The space $W(L^\infty, L^1)$ is sometimes called the *Wiener algebra* (although this term is sometimes used to denote $\mathcal{F}L^1$), cf. [51]. It was shown in [59] that $W(L^\infty, L^1)$ is a convenient and general class of windows for Gabor analysis within L^2.

Since any cube Q_α in $\mathbb{R}^d$ can be covered by a finite number of translates of a cube Q_β, the space $W(L^p, L^q_\nu)$ is independent of the value of α used in (7) in the sense that each different choice of α yields an equivalent norm for $W(L^p, L^q_\nu)$. A wide variety of other equivalent norms is provided by Feichtinger's theory of amalgam spaces [20, 19, 22]. We refer to [44] for an exposition of the "continuous" norms on the amalgam spaces.

The following lemma provides some useful inclusions among the amalgam spaces.

Lemma 3.1.1. *For each ω-moderate weight ν, we have the following inclusion relations: if $p_1 \ge p_2$, and $q_1 \le q_2$, then*

$$W(L^{p_1}, L^{q_1}_\omega) \subset W(L^{p_1}, L^{q_1}_\nu) \subset W(L^{p_2}, L^{q_2}_\nu) \subset W(L^{p_2}, L^{q_2}_{1/\omega}).$$

In particular, the inclusions $W(L^\infty, L^1_\omega) \subset W(L^p, L^q_\nu) \subset W(L^1, L^\infty_{1/\omega})$ *hold for all* $1 \leq p$, $q \leq \infty$ *and all* ω*-moderate weights* ν*. In this sense* $W(L^\infty, L^1_\omega)$ *is the smallest and* $W(L^1, L^\infty_{1/\omega})$ *is the largest amalgam space in the class of amalgam spaces with* ω*-moderate weight functions.*

Proof. The fact that ν is ω-moderate implies in particular that $\nu(x) \leq C\omega(x)$ for some positive constant C (this follows immediately from (1). Hence the inclusion $W(L^{p_1}, L^{q_1}_\omega) \subset W(L^{p_1}, L^{q_1}_\nu)$ follows from the (7).

Now let $f \in W(L^{p_1}, L^{q_1}_\nu)$. Then

$$\begin{aligned}
\|f\|_{W(L^{p_2}, L^{q_2}_\nu)} &= \left(\sum_{k \in \mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_{p_2}^{q_2} \, \nu(\alpha k)^{q_2} \right)^{1/q_2} \\
&\leq C \left(\sum_{k \in \mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_{p_1}^{q_2} \, \nu(\alpha k)^{q_2} \right)^{1/q_2} \\
&\leq C \left(\sum_{k \in \mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_{p_1}^{q_1} \, \nu(\alpha k)^{q_1} \right)^{1/q_1},
\end{aligned}$$

where we have used the inclusions $\ell^{q_1}(\mathbb{Z}^d) \subset \ell^{q_2}(\mathbb{Z}^d)$ (because $q_1 \leq q_2$), as well as $L^{p_1}(K) \subset L^{p_2}(K)$ for $p_1 \geq p_2$ and K a compact subset of $\mathbb{R}^d$. Thus we obtain $W(L^{p_1}, L^{q_1}_\nu) \subset W(L^{p_2}, L^{q_2}_\nu)$.

The last inclusion, $W(L^{p_2}, L^{q_2}_\nu) \subset W(L^{p_2}, L^{q_2}_{1/\omega})$ follows again from the fact that ν is ω-moderate, and so is $1/\nu$, and so $\frac{1}{\nu(x)} \leq C\omega(x)$ for all $x \in \mathbb{R}^d$.

The last part of the lemma is just an application of the above with $p_1 = \infty, p_2 = p, q_1 = 1$, and $q_2 = q$. □

Remark 3.1.2. For p, $q < \infty$, the Schwartz class $\mathcal{S}$ and the space of functions with compact support are dense in $W(L^p, L^q_\nu)$.

3.1.2 Duality and convergence

We will need to be precise about the meaning of convergence of series. For general references we refer to the text of Singer [54], and for references on Banach function spaces we refer to the text of Bennett and Sharpley [5].

The following lemma characterizing unconditional convergence will be useful.

Lemma 3.1.3. *Let X be a Banach space with dual space X^*, and let $f_k \in X$ for $k \in J$. Then the following statements are equivalent.*

a. $\sum_{k\in J} f_k$ *converges unconditionally in X, i.e., it converges with respect to every ordering of the index set J.*

b. *There exists $f \in X$ such that for each $\varepsilon > 0$, there exists a finite $F_0 \subset J$ such that*

$$\forall \text{ finite } F \supset F_0, \quad \Big\| f - \sum_{k\in F} f_k \Big\|_X \; < \; \varepsilon.$$

c. *For every $\varepsilon > 0$, there exists a finite $F_0 \subset J$ such that*

$$\forall \text{ finite } F \supset F_0, \quad \sup\Big\{ \sum_{k\notin F} |\langle f_k, h\rangle| : h \in X^*,\ \|h\|_{X^*} = 1 \Big\} \; < \; \varepsilon.$$

Now let X be a Banach function space in the sense of [5]. In particular, this includes the amalgam spaces $W(L^p, L^q_\nu)$. The *Köthe dual* of X (or the *associated space*, as it is called in [5]), is the space $\tilde{X}$ consisting of all measurable functions h such that $fh \in L^1$ for each $f \in X$. By [5, Thm. 1.2.9], $\tilde{X}$ is a closed, norm-fundamental subspace of X^*, so in particular,

$$\forall f \in X, \quad \|f\|_X \;=\; \sup\big\{ |\langle f, h\rangle| : h \in \tilde{X},\ \|h\|_{\tilde{X}} = 1 \big\}.$$

By [5, Cor. 1.5.3], X is complete in the $\sigma(X, \tilde{X})$ topology, i.e., the weak topology on X generated by $\tilde{X}$. In particular, a series $\sum_{k\in J} f_k$ converges in the $\sigma(X, \tilde{X})$ topology if $\sum_{k\in J} \langle f_k, h\rangle$ converges for each $h \in \tilde{X}$. It converges unconditionally in that topology if the convergence is independent of the ordering of J, and since the terms $\langle f_k, h\rangle$ are scalars, this occurs if and only if

$$\forall\, h \in \tilde{X}, \quad \sum_{k\in J} |\langle f_k, h\rangle| \;<\; \infty.$$

Remark 3.1.4. a. If X_1 and X_2 are two Banach function spaces such that $X_1 \subset X_2$ then $\tilde{X_2} \subset \tilde{X_1}$. Indeed, let $f \in \tilde{X_2}$ and $g \in X_1$ with $\|g\|_{X_1} = 1$, then by the definition of the Köthe dual, we have that $f\bar{g} \in L^1$, which implies that $f \in \tilde{X_1}$, and, moreover,

$$\begin{aligned} |\langle f, g\rangle| &= \Big|\int_{\mathbb{R}^d} f(x)\,\overline{g(x)}\,dx\Big| \\ &\le \|f\|_{\tilde{X_2}}\,\|g\|_{X_1} \\ &\le \|f\|_{\tilde{X_2}}. \end{aligned}$$

Thus,

$$\|f\|_{\tilde{X_1}} \le \|f\|_{\tilde{X_2}}. \tag{8}$$

b. For every Banach function space X we have $\tilde{\tilde{X}} = X$, cf. [5, Theorem 2.7].

The dual and Köthe dual of the amalgam spaces are given in the next lemma.

Lemma 3.1.5. *Let ν be an ω-moderate weight.*

a. *For $1 \le p, q < \infty$, the dual space of $W(L^p, L^q_\nu)$ is $W(L^{p'}, L^{q'}_{1/\nu})$.*

b. *For $1 \le p, q \le \infty$, the Köthe dual of $W(L^p, L^q_\nu)$ is $W(L^{p'}, L^{q'}_{1/\nu})$.*

Proof. a. We refer to [28, 19] for the proof of this part.

b. If $1 \le p, q < \infty$, the result follows from part a. Now assume that $p = \infty$ or $q = \infty$. We divide the proof in three parts.

Case I: $1 \leq p < \infty$ and $q = \infty$. Let $f \in W(L^{p'}, L^1_{1/\nu})$, and $g \in W(L^p, L^\infty_\nu)$ with $\|g\|_{W(L^p, L^\infty_\nu)} = 1$. Then we have

$$
\begin{aligned}
|\langle f, g\rangle| &= \Big|\int_{\mathbb{R}^d} f(x)\,\overline{g(x)}\,dx\Big| \\
&\leq \int_{\mathbb{R}^d} |f(x)|\,|g(x)|\,dx \\
&= \sum_{k\in\mathbb{Z}^d} \int_{\alpha k + Q_\alpha} |f(x)|\,|g(x)|\,dx \\
&\leq \sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_{p'}\, \|g \cdot T_{\alpha k}\chi_{Q_\alpha}\|_p \\
&= \sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_{p'}\, \frac{1}{\nu(\alpha k)}\, \nu(\alpha k)\, \|g \cdot T_{\alpha k}\chi_{Q_\alpha}\|_p \\
&\leq \sup_{k\in\mathbb{Z}^d} \|g \cdot T_{\alpha k}\chi_{Q_\alpha}\|_p\, \nu(\alpha k) \sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_{p'}\, \frac{1}{\nu(\alpha k)} \\
&= \|g\|_{W(L^p, L^\infty_\nu)}\, \|f\|_{W(L^{p'}, L^1_{1/\nu})} \\
&= \|f\|_{W(L^{p'}, L^1_{1/\nu})}.
\end{aligned}
$$

Thus,

$$\|f\|_{\tilde{W}(L^p, L^\infty_\nu)} \leq \|f\|_{W(L^{p'}, L^1_{1/\nu})},$$

and consequently, $W(L^{p'}, L^1_{1/\nu}) \subset \tilde{W}(L^p, L^\infty_\nu)$. From part a and (8), as well as Remark 3.1.4, we obtain the reverse inclusion, which completes the proof in this case.

Case II: if $p = \infty$ and $1 \leq q < \infty$ then the proof is very similar to the above so we omit it.

Case III: if $p = q = \infty$, we easily see that $L^1_{1/\nu} \subset (L^\infty_\nu)\tilde{}$, and the proof follows from the same arguments as above.

□

3.1.3 Sequence spaces

Before stating our results, we must define the sequence spaces that will be associated with Gabor expansions in the amalgam spaces. We begin by recalling that the Fourier transform of $f \in L^1(Q_{1/\beta})$ is the sequence $\hat{f}$ defined by

$$\hat{f}(n) \;=\; \mathcal{F}f(n) \;=\; \beta^d \int_{Q_{1/\beta}} f(t)\, e^{-2\pi i \beta n \cdot t}\, dt, \qquad n \in \mathbb{Z}^d.$$

For $1 \le p \le \infty$, let $\mathcal{F}L^p(Q_{1/\beta})$ denote the image of $L^p(Q_{1/\beta})$ under the Fourier transform. Since Fourier coefficients are unique in L^p, if $c = (c_n)_{n \in \mathbb{Z}^d} \in \mathcal{F}L^p(Q_{1/\beta})$ then there exists a unique function $m \in L^p(Q_{1/\beta})$ such that $\hat{m}(n) = c_n$ for every n, and the norm on $\mathcal{F}L^p(Q_{1/\beta})$ is defined by

$$\|c\|_{\mathcal{F}L^p(Q_{1/\beta})} \;=\; \|m\|_{p,Q_{1/\beta}}. \tag{9}$$

For $1 < p < \infty$, Littlewood–Paley theory can be used to give an equivalent norm for (9), cf. [16, Ch. 7]. The ongoing development motivates the following definition.

Definition 3.1.6. Let α, $\beta > 0$ be given. Then $S^{p,q}_{\tilde{\nu}} = \ell^q_{\tilde{\nu}}(\mathcal{F}L^p(Q_{1/\beta}))$ will denote the space of all $\mathcal{F}L^p(Q_{1/\beta})$-valued sequences which are $\ell^q_{\tilde{\nu}}$-summable. That is, a doubly-indexed sequence $c = (c_{kn})_{k,n \in \mathbb{Z}^d}$ lies in $S^{p,q}_{\tilde{\nu}}$ if for each $k \in \mathbb{Z}^d$ there exists $m_k \in L^p(Q_{1/\beta})$ such that

$$\hat{m}_k(n) \;=\; c_{kn}, \qquad k, n \in \mathbb{Z}^d,$$

and such that

$$\|c\|_{S^{p,q}_{\tilde{\nu}}} \;=\; \Bigl(\sum_{k \in \mathbb{Z}^d} \|m_k\|^q_{p,Q_{1/\beta}}\, \tilde{\nu}(k)^q\Bigr)^{1/q} \;<\; \infty,$$

with the usual change if $q = \infty$.

When $1 < p < \infty$, we can write m_k as a Fourier series

$$m_k(x) \;=\; \sum_{n \in \mathbb{Z}^d} c_{kn}\, e^{2\pi i \beta n \cdot x}, \tag{10}$$

in the sense that the square partial sums of (10) converge to m_k in the norm of $L^p(Q_{1/\beta})$, cf. [48], [63]. Hence, for $1 < p < \infty$ and $1 \le q < \infty$ we can write the norm on $S^{p,q}_{\tilde{\nu}}$ as

$$\|c\|_{S^{p,q}_{\tilde{\nu}}} \;=\; \biggl(\sum_{k\in\mathbb{Z}^d}\biggl(\int_{Q_{1/\beta}}\biggl|\sum_{n\in\mathbb{Z}^d} c_{kn}\, e^{2\pi i\beta n\cdot x}\biggr|^p\, dx\biggr)^{q/p} \tilde{\nu}(k)^q\biggr)^{1/q}.$$

Remark 3.1.7. A Banach function space X is called a solid space if $f \in X$ and $|g| \le |f|$ implies that $g \in X$ and, moreover, $\|g\| \le \|f\|$.

Note that for $p = 2$, we have via the Plancherel theorem that $S^{2,q}_{\tilde{\nu}} = \ell^{2,q}_{\tilde{\nu}}$, thus is a solid space. However, for general $p \ne 2$, $S^{p,q}_{\tilde{\nu}}$ is not a solid space. In particular, changing the phases of the c_{kn} can change the norm of c.

3.2 Boundedness of the analysis and synthesis operators

In this section, we prove the boundedness of the analysis and synthesis operators on the amalgam spaces. Moreover, we show that the Walnut representation, which is an extremely useful tool in Gabor analysis, holds on the amalgam spaces. However, before presenting these results, we give here some Lemmas that will be needed in the sequel.

3.2.1 Lemmas

The following lemmas will be important in the sequel. The first lemma is simply a counting argument.

Lemma 3.2.1. *Let α, $\beta > 0$ be given. Let $K_{\alpha\beta}$ be the maximum number of $\frac{1}{\beta}\mathbb{Z}^d$-translates of $Q_{1/\beta}$ required to cover any $\alpha\mathbb{Z}^d$-translate of Q_α, i.e.,*

$$K_{\alpha\beta} = \max_{k\in\mathbb{Z}^d} \#\bigl\{\ell \in \mathbb{Z}^d : \bigl|(\tfrac{\ell}{\beta} + Q_{1/\beta}) \cap (\alpha k + Q_\alpha)\bigr| > 0\bigr\}.$$

Then given $1 \le p \le \infty$, we have for any $1/\beta$-periodic function $m \in L^p(Q_{1/\beta})$ and any $k \in \mathbb{Z}^d$ that

$$\|m\|_{p,\alpha k+Q_\alpha} \le K_{\alpha\beta}^{1/p} \|m\|_{p,Q_{1/\beta}},$$

where $K_{\alpha\beta}^{1/\infty} = 1$.

Proof. Let m be a $1/\beta$-periodic function in $L^p(Q_{1/\beta})$, where $1 \le p < \infty$. For any $k \in \mathbb{Z}^d$ define $A_k = \{l \in \mathbb{Z}^d : |(\frac{l}{\beta} + Q_{1/\beta}) \cap (\alpha k + Q_\alpha)| > 0\}$. Then we have:

$$\begin{aligned}
\|m\|_{p,\alpha k+Q_\alpha}^p &= \int_{\alpha k+Q_\alpha} |m(x)|^p \, dx \\
&= \sum_{l\in\mathbb{Z}^d} \int_{\frac{l}{\beta}+Q_{1/\beta}} |m(x)|^p \, T_{\alpha k}\chi_{Q_\alpha}(x) \, dx \\
&= \sum_{l\in A_k} \int_{\frac{l}{\beta}+Q_{1/\beta}} |m(x)|^p \, T_{\alpha k}\chi_{Q_\alpha}(x) \, dx \\
&\le \sum_{l\in A_k} \int_{\frac{l}{\beta}+Q_{1/\beta}} |m(x)|^p \, dx \\
&= \sum_{l\in A_k} \int_{Q_{1/\beta}} |m(x)|^p \, dx \\
&\le K_{\alpha,\beta} \int_{Q_{1/\beta}} |m(x)|^p \, dx \\
&= K_{\alpha,\beta} \|m\|_{p,Q_{1/\beta}}^p.
\end{aligned}$$

If $p = \infty$, then it is easily seen that

$$\|m\|_{\infty,\alpha k+Q_\alpha} \le \|m\|_{\infty,Q_{1/\beta}}.$$

□

The second lemma is a weighted version of an estimate that is useful in the Walnut representation of the Gabor frame operator on L^2, see [59, Lemma 2.2].

Lemma 3.2.2. *Let ω be a submultiplicative weight, and let α, $\beta > 0$ be given. Then there exists a constant $C = C(\alpha, \beta, \omega) > 0$ such that if g, $\gamma \in W(L^\infty, L^1_\omega)$ and the* *functions G_n are defined by (22), then*

$$\sum_{n\in\mathbb{Z}^d} \|G_n\|_\infty\, \omega(\tfrac{n}{\beta}) \le C\, \|g\|_{W(L^\infty,L^1_\omega)}\, \|\gamma\|_{W(L^\infty,L^1_\omega)}.$$

Proof. It follows from the fact that ω is ω-moderate that $\|f\omega\|_{W(L^\infty,L^1)}$ is an equivalent norm for $W(L^\infty, L^1_\omega)$. In particular, we have $g\omega$, $\gamma\omega \in W(L^\infty, L^1)$, so by [41, Lemma 6.3.1],

$$\sum_{n\in\mathbb{Z}^d} \|\tilde{G}_n\|_\infty \le \left(\tfrac{1}{\alpha}+1\right)^d (2\beta+1)^d\, \|g\omega\|_{W(L^\infty,L^1)}\, \|\gamma\omega\|_{W(L^\infty,L^1)},$$

where $\tilde{G}_n$ is the analogue of G_n with g replaced by $|g|\omega$ and γ replaced by $|\gamma|\omega$. Hence,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^d} \|G_n\|_\infty\, \omega(\tfrac{n}{\beta}) &= \sum_{n\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \Bigg| \sum_{k\in\mathbb{Z}^d} \overline{g(x-\tfrac{n}{\beta}-\alpha k)}\, \gamma(x-\alpha k)\times \\
&\qquad \omega\big((x-\alpha k)-(x-\tfrac{n}{\beta}-\alpha k)\big)\Bigg| \\
&\le \sum_{n\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} |g(x-\tfrac{n}{\beta}-\alpha k)|\, \omega(x-\tfrac{n}{\beta}-\alpha k)\times \\
&\qquad |\gamma(x-\alpha k)|\, \omega(x-\alpha k) \\
&= \sum_{n\in\mathbb{Z}^d} \|\tilde{G}_n\|_\infty \\
&\le C\, \|g\|_{W(L^\infty,L^1_\omega)}\, \|\gamma\|_{W(L^\infty,L^1_\omega)}.
\end{aligned}$$

□

Finally, we need an estimate on the effect of translations on the amalgam space norm.

Lemma 3.2.3. *Let ν be an ω-moderate weight. Then for $1 \le p, q \le \infty$, we have for each $f \in W(L^p, L^q_\nu)$ and $\ell \in \mathbb{Z}^d$ that*

$$\|T_{\alpha\ell} f\|_{W(L^p,L^q_\nu)} \;\le\; C_\nu\, \omega(\alpha\ell)\, \|f\|_{W(L^p,L^q_\nu)}.$$

Proof. Let $f \in W(L^p, L^q_\nu)$, and $l \in \mathbb{Z}^d$. Then using the fact that ν is ω-moderate, we obtain

$$\begin{aligned}
\|T_{\alpha l} f\|_{W(L^p,L^q_\nu)} &= \left(\sum_{k\in\mathbb{Z}^d} \|T_{\alpha l} f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_p^q \, \nu(\alpha k)^q \right)^{1/q} \\
&= \left(\sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha(k-l)}\chi_{Q_\alpha}\|_p^q \, \nu(\alpha k)^q \right)^{1/q} \\
&= \left(\sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_p^q \, \nu(\alpha(k+l))^q \right)^{1/q} \\
&\le C\,\omega(\alpha l) \left(\sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha k}\chi_{Q_\alpha}\|_p^q \, \nu(\alpha k)^q \right)^{1/q} \\
&= C\,\omega(\alpha l)\, \|f\|_{W(L^p,L^q_\nu)}.
\end{aligned}$$

□

3.2.2 Boundedness of the synthesis operator

Theorem 3.2.4. *Let ν be an ω-moderate weight on $\mathbb{R}^d$. Let α, $\beta > 0$ and $1 \le p, q \le \infty$ be given. Fix g, $\gamma \in W(L^\infty, L^1_\omega)$. Then the following statement is true. Given $c \in S^{p,q}_{\tilde{\nu}}$, let $m_k \in L^p(Q_\alpha)$ be the unique functions satisfying $\hat{m}_k(n) = c_{kn}$ for all k, $n \in \mathbb{Z}^d$. Then the series*

$$R_g c \;=\; \sum_{k\in\mathbb{Z}^d} m_k \cdot T_{\alpha k} g \tag{11}$$

converges unconditionally in $W(L^p, L^q_\nu)$ (unconditionally in the $\sigma(W(L^p, L^q_\nu), W(L^{p'}, L^{q'}_{1/\nu}))$ topology if $p = \infty$ or $q = \infty$), and R_g is a bounded mapping from $S^{p,q}_{\tilde{\nu}}$ into $W(L^p, L^q_\nu)$.

Proof. We divide the proof into cases. First, we consider the case $1 \le p, q < \infty$. We are given $c \in S^{p,q}_{\tilde{\nu}}$, and we must prove that the series (11) defining $R_g c$ converges unconditionally in the norm of $W(L^p, L^q_\nu)$, and that R_g so defined is a bounded mapping from $S^{p,q}_{\tilde{\nu}}$ into $W(L^p, L^q_\nu)$. To show the convergence we will make use of Lemma 3.1.3.

Fix $\varepsilon > 0$. Then, by definition of the norm in $S^{p,q}_{\tilde{\nu}}$, we have that

$$\sum \|m_k\|^q_{p,Q_{1/\beta}}\, \tilde{\nu}(k)^q < \infty.$$

Hence there exists a finite set F_0 such that

$$\forall \text{ finite } F \supset F_0, \quad \sum_{k \notin F} \|m_k\|^q_{p,Q_{1/\beta}}\, \tilde{\nu}(k)^q \;<\; \varepsilon^q. \tag{12}$$

Recall that $1/\nu$ is an ω-moderate weight, and let $K_{\alpha\beta}$ be the constant appearing in Lemma 3.2.1. Fix any $h \in W(L^{p'}, L^{q'}_{1/\nu})$. Then

$$\begin{aligned}
\sum_{k\notin F} \big|\big\langle m_k \cdot T_{\alpha k} g,\ h\big\rangle\big| &\le \sum_{k \notin F} \int_{\mathbb{R}^d} |m_k(x)\, T_{\alpha k} g(x)\, h(x)|\, dx \\
&= \sum_{k\notin F} \sum_{n \in \mathbb{Z}^d} \int_{Q_\alpha} |m_k(x)\, T_{\alpha k} g(x)\, h(x)|\, T_{\alpha n + \alpha k} \chi_{Q_\alpha}(x)\, dx \\
&\le \sum_{k\notin F} \sum_{n \in \mathbb{Z}^d} \|T_{\alpha k} g \cdot T_{\alpha n+\alpha k}\chi_{Q_\alpha}\|_\infty \, \|m_k\|_{p,\alpha n+\alpha k + Q_\alpha} \times \\
&\qquad \|h \cdot T_{\alpha n+\alpha k}\chi_{Q_\alpha}\|_{p'}\, \frac{\nu(\alpha k)}{\nu(\alpha n+\alpha k-\alpha n)} \\
&\le \sum_{n\in\mathbb{Z}^d} \|g \cdot T_{\alpha n}\chi_{Q_\alpha}\|_\infty \times \\
&\qquad \sum_{k\notin F} K^{1/p}_{\alpha\beta}\, \|m_k\|_{p,Q_{1/\beta}}\, \|h\cdot T_{\alpha n+\alpha k}\chi_{Q_\alpha}\|_{p'}\, \frac{C_\nu\, \nu(\alpha k)\, \omega(\alpha n)}{\nu(\alpha n+\alpha k)} \\
&\le C_\nu K^{1/p}_{\alpha\beta} \sum_{n\in\mathbb{Z}^d} \|g \cdot T_{\alpha n}\chi_{Q_\alpha}\|_\infty\, \omega(\alpha n) \times \\
&\qquad \Bigg(\sum_{k\notin F} \|m_k\|^q_{p,Q_{1/\beta}}\, \nu(\alpha k)^q\Bigg)^{1/q} \times \\
&\qquad \Bigg(\sum_{k\in\mathbb{Z}^d} \|h\cdot T_{\alpha n+\alpha k}\chi_{Q_\alpha}\|^{q'}_{p'}\, \frac{1}{\nu(\alpha n+\alpha k)^{q'}}\Bigg)^{1/q'}.
\end{aligned} \tag{13}$$

Combining (12) and (13), we have that

$$\sum_{k\notin F} \big|\big\langle m_k \cdot T_{\alpha k} g,\ h\big\rangle\big| \;\le\; \varepsilon C_\nu K^{1/p}_{\alpha\beta}\, \|g\|_{W(L^\infty, L^1_\omega)}\, \|h\|_{W(L^{p'}, L^{q'}_{1/\nu})}.$$

Therefore, taking the supremum over all h of unit norm and appealing to Lemma 3.1.3, we see that $R_g c = \sum m_k \cdot T_{\alpha k} g$ converges unconditionally. Further, replacing F by $\mathbb{Z}^d$ in the calculation in (13) yields

$$\begin{aligned} |\langle R_g c, h\rangle| &\le \sum_{k\in\mathbb{Z}^d} \bigl|\langle m_k \cdot T_{\alpha k} g,\ h\rangle\bigr| \\ &\le C_\nu K_{\alpha\beta}^{1/p}\, \|g\|_{W(L^\infty,L^1_\omega)}\, \|c\|_{S^{p,q}_\nu}\, \|h\|_{W(L^{p'},L^{q'}_{1/\nu})}. \end{aligned} \tag{14}$$

Since $W(L^{p'}, L^{q'}_{1/\nu})$ is the dual space of $W(L^p, L^q_\nu)$, taking the suprema over all h of unit norm in (14) shows that

$$\begin{aligned} \|R_g c\|_{W(L^p,L^q_\nu)} &= \sup\bigl\{|\langle R_g c, h\rangle| : \|h\|_{W(L^{p'},L^{q'}_{1/\nu})} = 1\bigr\} \\ &\le C_\nu K_{\alpha\beta}^{1/p}\, \|g\|_{W(L^\infty,L^1_\omega)}\, \|c\|_{S^{p,q}_\nu}, \end{aligned} \tag{15}$$

so R_g is bounded. This completes the proof for the case $1 \le p, q < \infty$.

When $p = \infty$ or $q = \infty$, we make use of the fact that $W(L^{p'}, L^{q'}_{1/\nu})$ is the Köthe dual of $W(L^p, L^q_\nu)$. The fact that the series defining $R_g c$ converges in the weak topology is given by the same calculations as in (13), (14), and the fact that the Köthe dual is a norm-fundamental subspace of the dual space means that we can again estimate $\|R_g c\|_{S^{p,q}_\nu}$ by using (15). Hence R_g is bounded, and the proof is complete. □

Remark 3.2.5. When $1 < p < \infty$, the functions m_k appearing in (11) can be written as Fourier series, allowing $R_g c$ to be written as the iterated sum

$$R_g c(x) \;=\; \sum_{k\in\mathbb{Z}^d} \Biggl(\sum_{n\in\mathbb{Z}^d} c_{kn}\, e^{2\pi i\beta n\cdot x}\Biggr)\, T_{\alpha k} g(x), \tag{16}$$

i.e., the same series as appears in the Gabor expansions in (3), or more generally the Gabor expansions in modulation spaces (6). When $p = 1$ or $p = \infty$, this is not the case. The functions m_k are still uniquely determined by c, but cannot be written as Fourier series. When $p = q = 2$, both the inner and outer sums in the iterated series in (16) converge unconditionally, and then $R_g c$ can also be written as the double sum given by (6), with unconditional convergence of that series.

3.2.3 Boundedness of the analysis operator

Theorem 3.2.6. *Let ν be an ω-moderate weight on $\mathbb{R}^d$. Let α, $\beta > 0$ and $1 \leq p, q \leq \infty$ be given. Fix g, $\gamma \in W(L^\infty, L^1_\omega)$. Then the analysis operator defined by $C_g f = \big(\langle f, M_{\beta n} T_{\alpha k} g\rangle\big)_{k,n\in\mathbb{Z}^d}$ is a bounded mapping from $W(L^p, L^q_\nu)$ into $S^{p,q}_{\tilde{\nu}}$, Moreover, there exist unique functions $m_k \in L^p(Q_{1/\beta})$ which satisfy $\hat{m}_k(n) = C_g f(k,n)$ for all k, $n \in \mathbb{Z}^d$, and these are given explicitly by*

$$m_k(x) = \beta^{-d} \sum_{n\in\mathbb{Z}^d} \big(f \cdot T_{\alpha k}\bar{g}\big)(x - \tfrac{n}{\beta})$$

$$= \beta^{-d} \sum_{n\in\mathbb{Z}^d} \big(T_{\frac{n}{\beta}} f \cdot T_{\alpha k + \frac{n}{\beta}}\bar{g}\big)(x). \tag{17}$$

The series on the right side of (17) converges unconditionally in $L^p(Q_{1/\beta})$ (unconditionally in the $\sigma(L^\infty(Q_{1/\beta}), L^1(Q_{1/\beta})$ topology if $p = \infty$).

Proof. We are given that $g \in W(L^\infty, L^1_\omega)$ and that $1 \leq p, q \leq \infty$. Let $f \in W(L^p, L^q_\nu)$, which is a subspace of $W(L^1, L^\infty_{1/\omega})$. First we must show that the functions m_k given by (17) are well-defined. Since m_k is the $1/\beta$-periodization of the integrable function $f \cdot T_{\alpha k}\overline{g}$, the series defining m_k converges at least in $L^1(Q_{1/\beta})$. To show that the periodization converges unconditionally in $L^p(Q_{1/\beta})$ (weakly if $p = \infty$) and to derive a useful estimate, fix any $1/\beta$-periodic function $h \in L^{p'}(Q_{1/\beta})$. Then for each fixed k, we have

$$\Big|\int_{Q_{1/\beta}} \sum_{n\in\mathbb{Z}^d} f(x - \tfrac{n}{\beta})\, T_{\alpha k}\overline{g}(x - \tfrac{n}{\beta})\, h(x)\, dx\Big|$$

$$\leq \int_{\mathbb{R}^d} |f(x)\, T_{\alpha k}\overline{g}(x)\, h(x)|\, dx$$

$$= \sum_{n\in\mathbb{Z}^d} \int_{Q_\alpha} |f(x)\, T_{\alpha k} g(x)\, h(x)|\, T_{\alpha k + \alpha n}\chi_{Q_\alpha}(x)\, dx$$

$$\leq \sum_{n\in\mathbb{Z}^d} \|T_{\alpha k}\overline{g} \cdot T_{\alpha k + \alpha n}\chi_{Q_\alpha}\|_\infty\, \|f \cdot T_{\alpha k + \alpha n}\chi_{Q_\alpha}\|_p\ \times$$

$$\|h\|_{p',\alpha k+\alpha n+Q_\alpha} \, \frac{\nu(\alpha k+\alpha n-\alpha n)}{\nu(\alpha k)}$$

$$\leq \sum_{n\in\mathbb{Z}^d} \|g\cdot T_{\alpha n}\chi_{Q_\alpha}\|_\infty \, \|f\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p \, K_{\alpha\beta}^{1/p'} \times$$

$$\|h\|_{p',Q_{1/\beta}} \, \frac{C_\nu \nu(\alpha k+\alpha n)\,\omega(\alpha n)}{\nu(\alpha k)}$$

$$= C_\nu K_{\alpha\beta}^{1/p'} \|h\|_{p',Q_{1/\beta}} \, \frac{1}{\nu(\alpha k)} \sum_{n\in\mathbb{Z}^d} \|g\cdot T_{\alpha n}\chi_{Q_\alpha}\|_\infty \, \omega(\alpha n) \times$$

$$\|f\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p \, \nu(\alpha k+\alpha n). \tag{18}$$

This yields the desired convergence, and taking the suprema in (18) over h with unit norm implies the estimate

$$\|m_k\|_{p,Q_{1/\beta}} \leq \beta^{-d} C_\nu K_{\alpha\beta}^{1/p'} \tfrac{1}{\nu(\alpha k)} \textstyle\sum_{n\in\mathbb{Z}^d} \|g\cdot T_{\alpha n}\chi_{Q_\alpha}\|_\infty \, \omega(\alpha n) \times$$

$$\|f\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p \, \nu(\alpha k+\alpha n). \tag{19}$$

Second, we show that $\hat{m}_k(n)$ has the correct form. Since $e^{2\pi i\beta n\cdot x} \in L^{p'}(Q_{1/\beta})$, we have by the weak convergence of the series defining m_k that

$$\begin{aligned}
\hat{m}_k(n) &= \beta^d \left\langle m_k,\ e^{2\pi i\beta n\cdot x}\right\rangle \\
&= \sum_{\ell\in\mathbb{Z}^d} \int_{Q_{1/\beta}} \left\langle T_{\frac{\ell}{\beta}} f\cdot T_{\alpha k+\frac{\ell}{\beta}}\bar{g},\ e^{2\pi i\beta n\cdot x}\right\rangle \\
&= \sum_{\ell\in\mathbb{Z}^d} \int_{Q_{1/\beta}} f(x-\tfrac{\ell}{\beta})\, T_{\alpha k}\bar{g}(x-\tfrac{\ell}{\beta})\, e^{-2\pi i\beta n\cdot(x-\ell/\beta)}\, dx \\
&= \int_{\mathbb{R}^d} \big(f\cdot T_{\alpha k}\bar{g}\big)(x)\, e^{-2\pi i\beta n\cdot x}\, dx \\
&= \left\langle f,\ M_{\beta n} T_{\alpha k} g\right\rangle \\
&= C_g f(k,n).
\end{aligned}$$

Finally, we must show that C_g is a bounded mapping of $W(L^p, L^q_\nu)$ into $S^{p,q}_{\tilde{\nu}}$. Given $f \in W(L^p, L^q_\nu)$, to show that $C_g f \in S^{p,q}_{\tilde{\nu}}$ we must show that the sequence r given by

$$r(k) = \|m_k\|_{p,Q_{1/\beta}}, \qquad k\in\mathbb{Z}^d,$$

lies in $\ell^q_{\tilde\nu}$. To do this, fix any sequence $a \in \ell^{q'}_{1/\tilde\nu}$. Then, using (19), we have

$$\begin{aligned}
|\langle r, a\rangle| &\le \sum_{k\in\mathbb{Z}^d} \|m_k\|_{p,Q_{1/\beta}}\, |a(k)| \\
&\le \beta^{-d} C_\nu K_{\alpha\beta}^{1/p'} \sum_{n\in\mathbb{Z}^d} \|g\cdot T_{\alpha n}\chi_{Q_\alpha}\|_\infty\, \omega(\alpha n) \times \\
&\qquad \sum_{k\in\mathbb{Z}^d} \|f\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p\, \nu(\alpha k+\alpha n)\, |a(k)|\, \frac{1}{\nu(\alpha k)} \\
&\le \beta^{-d} C_\nu K_{\alpha\beta}^{1/p'} \sum_{n\in\mathbb{Z}^d} \|g\cdot T_{\alpha n}\chi_{Q_\alpha}\|_\infty\, \omega(\alpha n) \times \\
&\qquad \Bigg(\sum_{k\in\mathbb{Z}^d} \|f\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p^q\, \nu(\alpha k+\alpha n)^q\Bigg)^{1/q} \times \\
&\qquad \Bigg(\sum_{k\in\mathbb{Z}^d} |a(k)|^{q'}\, \frac{1}{\nu(\alpha k)^{q'}}\Bigg)^{1/q'} \\
&\le \beta^{-d} C_\nu K_{\alpha\beta}^{1/p'}\, \|g\|_{W(L^\infty,L^1_\omega)}\, \|f\|_{W(L^p,L^q_\nu)}\, \|a\|_{\ell^{q'}_{1/\tilde\nu}}.
\end{aligned} \tag{20}$$

Since $\ell^{q'}_{1/\tilde\nu}$ equals $(\ell^q_{\tilde\nu})^*$ when $q<\infty$ and is a norm-fundamental subspace when $q=\infty$, taking the suprema in (20) over sequences a with unit norm yields the estimate

$$\|C_g f\|_{S^{p,q}_{\tilde\nu}} \;=\; \|r\|_{\ell^p_{\tilde\nu}} \;\le\; \beta^{-d} C_\nu K_{\alpha\beta}^{1/p'}\, \|g\|_{W(L^\infty,L^1_\omega)}\, \|f\|_{W(L^p,L^q_\nu)}.$$

Hence C_g is a bounded mapping of $W(L^p,L^q_\nu)$ into $S^{p,q}_{\tilde\nu}$. □

Remark 3.2.7. For the case $1<p,q<\infty$, the boundedness of C_g could also be shown by proving that $C_g\colon W(L^p,L^q_\nu)\to S^{p,q}_{\tilde\nu}$ is the adjoint of $R_g\colon S^{p',q'}_{1/\tilde\nu}\to W(L^{p'},L^{q'}_{1/\nu})$, and then using the reflexivity of the space $W(L^p,L^q_\nu)$ and the fact that $1/\nu$ is also ω-moderate.

3.2.4 The Walnut representation of the Gabor frame operator on amalgam spaces

Theorem 3.2.8. *Let ν be an ω-moderate weight on $\mathbb{R}^d$. Let α, $\beta > 0$ and $1 \le p, q \le \infty$ be given. Fix g, $\gamma \in W(L^\infty, L^1_\omega)$. Then the* Walnut representation

$$R_\gamma C_g f \;=\; \beta^{-d} \sum_{n \in \mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}} f \tag{21}$$

holds for $f \in W(L^p, L^q_\nu)$, with the series on the right of (21) *converging absolutely in $W(L^p, L^q_\nu)$, and where*

$$\begin{aligned} G_n(x) &= \sum_{k \in \mathbb{Z}^d} \overline{g(x - \tfrac{n}{\beta} - \alpha k)}\, \gamma(x - \alpha k) \\ &= \sum_{k \in \mathbb{Z}^d} \big(T_{\alpha k + \frac{n}{\beta}} \bar{g} \cdot T_{\alpha k} \gamma\big)(x). \end{aligned} \tag{22}$$

Proof. We are given g, $\gamma \in W(L^\infty, L^1_\omega)$ and $1 \le p, q \le \infty$. For this proof, let us use the equivalent norm for $W(L^p, L^q_\nu)$ obtained by replacing α in (7) by $1/\beta$. Then by Lemma 3.2.3,

$$\|T_{\frac{n}{\beta}} f\|_{W(L^p, L^q_\nu)} \;\le\; C_\nu\, \omega(\tfrac{n}{\beta})\, \|f\|_{W(L^p, L^q_\nu)}.$$

Therefore, using the autocorrelation functions G_n defined in (22), we have for $f \in W(L^p, L^q_\nu)$ that

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d} \|G_n \cdot T_{\frac{n}{\beta}} f\|_{W(L^p, L^q_\nu)} &\le \sum_{n \in \mathbb{Z}^d} \|G_n\|_\infty \, \|T_{\frac{n}{\beta}} f\|_{W(L^p, L^q_\nu)} \\ &\le C_\nu \, \|f\|_{W(L^p, L^q_\nu)} \sum_{n \in \mathbb{Z}^d} \|G_n\|_\infty \, \omega(\tfrac{n}{\beta}) \\ &\le C_\nu \, \|f\|_{W(L^p, L^q_\nu)} \, \|g\|_{W(L^\infty, L^1_\omega)} \, \|\gamma\|_{W(L^\infty, L^1_\omega)}, \end{aligned}$$

the last inequality following from Lemma 3.2.2. Hence the series $\sum G_n \cdot T_{\frac{n}{\beta}} f$ converges absolutely in $W(L^p, L^q_\nu)$.

Now fix $f \in W(L^p, L^q_\nu)$. Then $C_g f \in S^{p,q}_{\tilde{\nu}}$ by Theorem 3.2.6. Letting m_k be defined by (17), we have $C_g f(k, n) = \hat{m}_k(n)$. Further, $R_\gamma C_g f = \sum m_k \cdot T_{\alpha k} \gamma$, this series converging unconditionally if p, $q < \infty$, or unconditionally in the weak topology otherwise. In any case, for $h \in W(L^{p'}, L^{q'}_{1/\tilde{\nu}})$ we have

$$\begin{aligned}
\big\langle R_\gamma C_g f,\ h\big\rangle &= \sum_{k\in\mathbb{Z}^d} \big\langle m_k \cdot T_{\alpha k}\gamma,\ h\big\rangle \\
&= \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} m_k(x)\, T_{\alpha k}\gamma(x)\, \bar{h}(x)\, dx \\
&= \beta^{-d} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} T_{\frac{n}{\beta}} f(x)\, T_{\alpha k+\frac{n}{\beta}} \bar{g}(x)\, T_{\alpha k}\gamma(x)\, \bar{h}(x)\, dx \\
&= \beta^{-d} \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} T_{\frac{n}{\beta}} f(x)\, T_{\alpha k+\frac{n}{\beta}} \bar{g}(x)\, T_{\alpha k}\gamma(x)\, \bar{h}(x)\, dx \\
&= \beta^{-d} \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} T_{\frac{n}{\beta}} f(x)\, G_n(x)\, \bar{h}(x)\, dx. \\
&= \beta^{-d} \sum_{n\in\mathbb{Z}^d} \big\langle G_n \cdot T_{\frac{n}{\beta}} f,\ h\big\rangle,
\end{aligned}$$

from which (21) follows. The interchanges of integration and summation can be justified by Lemma 3.2.2 and Fubini's Theorem. □

3.3 Gabor expansions in the amalgam spaces

Under the assumption that $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$, we obtain the following result, which makes precise the characterization of the amalgam spaces in terms of Gabor frames. In particular, we show in this section that there is an analogue for the amalgam spaces of the Gabor expansions of functions in the modulation spaces (see Theorem 2.3.11). This is surprising, because the modulation spaces are the natural setting for Gabor analysis. And indeed, while Gabor expansions converge unconditionally in the modulation spaces, the convergence in the amalgam spaces is conditional in general and even the meaning of the term "expansion" must be handled appropriately. Throughout, we will use the notation $\tilde{\nu}(k) = \nu(\alpha k)$.

Theorem 3.3.1. *Let ν be an ω-moderate weight on $\mathbb{R}^d$, and let α, $\beta > 0$ and $1 \leq p, q \leq \infty$ be given. Assume that g, $\gamma \in W(L^\infty, L^1_\omega)$ are such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2 with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$. Then the following statements hold.*

a. $R_\gamma C_g = I$ *on* $W(L^p, L^q_\nu)$.

b. *We have the norm equivalence* $\|f\|_{W(L^p,L^q_\nu)} \asymp \|C_g f\|_{S^{p,q}_{\tilde{\nu}}}$.

c. *A function* $f \in W(L^1, L^\infty_{1/\omega})$ *belongs to* $W(L^p, L^q_\nu)$ *if and only if* $C_g f \in S^{p,q}_{\tilde{\nu}}$.

Proof. We are given g, $\gamma \in W(L^\infty, L^1_\omega)$ such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2 and γ is the dual window to g. By Theorem 3.2.6, we have that C_g, $C_\gamma \colon W(L^p, L^q_\nu) \to S^{p,q}_{\tilde{\nu}}$ and R_g, $R_\gamma \colon S^{p,q}_{\tilde{\nu}} \to W(L^p, L^q_\nu)$ are bounded mappings for each $1 \leq p, q \leq \infty$ and each ω-moderate weight ν. Further, for the case $p = q = 2$ and $\nu = 1$, the frame hypothesis implies that the identity $R_\gamma C_g = I$ holds on L^2, and the definition of R_γ given in Chapter 2 coincides in this case with the definition of R_γ given in Theorem 3.2.4. Letting G_n be the autocorrelation functions defined in (22), the fact that $R_\gamma C_g = I$ holds on L^2 implies by [41, Thm. 7.3.1] that

$$\beta^{-d} G_0 = 1 \text{ a.e.} \qquad \text{and} \qquad G_n = 0 \text{ a.e. for } n \neq 0.$$

Consequently, using the Walnut representation (21) of $R_\gamma C_g$ on the space $W(L^p, L^q_\nu)$, we have for $f \in W(L^p, L^q_\nu)$ that

$$R_\gamma C_g f \;=\; \beta^{-d} \sum_{n \in \mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}} f \;=\; f.$$

Hence $R_\gamma C_g = I$ holds on $W(L^p, L^q_\nu)$ as well. This proves part a of Theorem 3.3.1.

Next, given $f \in W(L^p, L^q_\nu)$, we have

$$\begin{aligned} \|f\|_{W(L^p,L^q_\nu)} &= \|R_\gamma C_g f\|_{W(L^p,L^q_\nu)} \\ &\leq \|R_\gamma\| \, \|C_g f\|_{S^{p,q}_{\tilde{\nu}}} \\ &\leq \|R_\gamma\| \, \|C_g\| \, \|f\|_{W(L^p,L^q_\nu)}. \end{aligned}$$

Consequently, $\|C_g f\|_{S^{p,q}_{\tilde{\nu}}} \asymp \|f\|_{W(L^p, L^q_\nu)}$, which proves part b of Theorem 3.3.1.

Finally, we prove part c of Theorem 3.3.1. Let $f \in W(L^1, L^\infty_{1/\omega})$ be given. We must show that $f \in W(L^p, L^q_\nu)$ if and only if $C_g f \in S^{p,q}_{\tilde{\nu}}$. The forward direction, that if $f \in W(L^p, L^q_\nu)$ then $C_g f \in S^{p,q}_{\tilde{\nu}}$, is simply Theorem 3.2.6. For the reverse direction, assume that $C_g f \in S^{p,q}_{\tilde{\nu}}$. Then by Theorem 3.2.6, the function $\tilde{f} = R_\gamma(C_g f)$ lies in $W(L^p, L^q_\nu)$. However, the factorization $R_\gamma C_g = I$ holds on every amalgam space, including $W(L^1, L^\infty_{1/\omega})$ in particular, so we also know that $f = R_\gamma C_g f$. Thus $f = \tilde{f} \in W(L^p, L^q_\nu)$, which completes the proof. □

Remark 3.3.2. a. Theorem 3.3.1 says that, given an appropriate condition on the window g and its dual window γ, a Gabor frame for L^2 extends to the amalgam spaces and provides "Gabor expansions" for the amalgam spaces in the sense that we have the factorization of the identity as $I = R_\gamma C_g$. The specific form of these expansions is that given f, there exist functions m_k such that $f = R_\gamma C_g f = \sum m_k \cdot T_{\alpha k} g$. When $1 < p < \infty$, the functions m_k can be realized as Fourier series, leading to an expansion of the form

$$f(x) \;=\; R_\gamma C_g f(x) \;=\; \sum_{k\in\mathbb{Z}^d} \left(\sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k}\gamma\rangle \, e^{2\pi i \beta n\cdot x} \right) T_{\alpha k} g(x). \tag{23}$$

The inner sum defining m_k converges conditionally in general, while the outer sum converges unconditionally.

b. For the case $p = 1$, the functions m_k cannot be written as Fourier series, so we do not have a series expansion of the form (23). A different approach to the case $p = q = 1$ and $\nu = 1$, based on Littlewood–Paley theory, is developed by Gilbert and Lakey in [33], where they show that Gabor frames can be used to characterize a Hardy-type space on the line.

c. Theorem 3.3.1c says that if we use the "largest" amalgam space $W(L^1, L^\infty_{1/\omega})$ as our "universe," then membership of a function in an amalgam $W(L^p, L^q_\nu)$ is *characterized* by membership of its sequence of Gabor coefficients in an appropriate sequence space. By imposing additional restrictions on g, γ, we could enlarge the universe on which this characterization is valid. In particular, if we required g, γ to lie in the Schwartz class $\mathcal{S}$, then the universe on which this characterization was valid would be the space $\mathcal{S}'$ of tempered distributions.

d. For the case of the modulation spaces, there is a deep result that states that if g lies in the Feichtinger algebra M^1_ω, then the dual window γ will lie in M^1_ω as well, [43]. For the case of the amalgam spaces, we do not know if the assumption $g \in W(L^\infty, L^1_\omega)$ implies that the dual window γ also lies in that space. This is an interesting and possibly difficult open question.

3.4 Convergence of Gabor expansions

As pointed out above, when $1 < p < \infty$, the synthesis operator R_g can be written as the iterated sum (16). The inner series in this sum converges conditionally in general, while the outer series converges unconditionally. Our next result shows that this series can also be written as a double sum as in Theorem 2.3.11, but because the proof relies on the convergence of Fourier series in L^p, the convergence is conditional in general. In dealing with Fourier series in higher dimensions, it is important to use the maximum norm $|x| = \max\{|x_1|, \dots, |x_d|\}$ on $\mathbb{R}^d$.

Theorem 3.4.1. *Let ν be an ω-moderate weight. Let α, $\beta > 0$ and $1 < p < \infty$, $1 \le q < \infty$ be given. Assume that g, $\gamma \in W(L^\infty, L^1_\omega)$ are such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2 with dual window γ. Then the following statements hold.*

a. *If $c \in S^{p,q}_\nu$, then the partial sums*

$$S_{K,N}c \;=\; \sum_{|k|\le K} \sum_{|n|\le N} c_{kn}\, M_{\beta n} T_{\alpha k} g, \qquad K, N > 0,$$

converge to $R_g c$ in the norm of $W(L^p, L^q_\nu)$, i.e., for each $\varepsilon > 0$ there exist K_0, $N_0 > 0$ such that

$$\forall\, K \ge K_0, \quad \forall\, N \ge N_0, \quad \|R_g c - S_{K,N}c\|_{W(L^p, L^q_\nu)} \;<\; \varepsilon.$$

b. *If* $f \in W(L^p, L^q_\nu)$*, then the partial sums of the Gabor expansion of* f*,*

$$S_{K,N}(C_g f) \;=\; \sum_{|k|\le K}\sum_{|n|\le N} \langle f, M_{\beta n}T_{\alpha k}g\rangle\, M_{\beta n}T_{\alpha k}\gamma,$$

converge to f *in the norm of* $W(L^p, L^q_\nu)$*.*

Proof. We are given $g,\, \gamma \in W(L^\infty, L^1_\omega)$ such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for L^2 and γ is the dual window to g, and we fix $1 < p < \infty$ and $1 \le q < \infty$.

Assume that $c \in S^{p,q}_{\tilde\nu}$, and let m_k be defined by (17). For $N > 0$, write

$$S_N m_k(x) \;=\; \sum_{|n|\le N} c_{kn}\, e^{2\pi i\beta n\cdot x}$$

for the partial sums of the Fourier series of m_k. The exponentials $\{e^{2\pi i\beta n\cdot x}\}_{n\in\mathbb{Z}^d}$ form a basis for $L^p(Q_{1/\beta})$ [48], [63], so, letting C_1 denote the basis constant for this system, we have for each $k \in \mathbb{Z}^d$ that

$$\lim_{N\to\infty} \|m_k - S_N m_k\|_{p,Q_{1/\beta}} \;=\; 0 \tag{24}$$

and

$$\sup_{N>0} \|S_N m_k\|_{p,Q_{1/\beta}} \;\le\; C_1\, \|m_k\|_{p,Q_{1/\beta}}. \tag{25}$$

Since $c \in S^{p,q}_{\tilde\nu}$, given $\varepsilon > 0$, we can find $K_0 > 0$ such that

$$\forall\, K \ge K_0, \quad \Biggl(\sum_{|k|\ge K} \|m_k\|^q_{p,Q_{1/\beta}}\, \tilde\nu(k)^q\Biggr)^{1/q} \;<\; \varepsilon. \tag{26}$$

Because of (24) and the fact that K_0 is finite, we can find an $N_0 > 0$ such that

$$\forall\, N \ge N_0, \quad \sup_{|k|\le K_0} \|m_k - S_N m_k\|_{p,Q_{1/\beta}}\, \tilde\nu(k) \;<\; \frac{\varepsilon}{(2K_0+1)^{d/q}}. \tag{27}$$

Now, since $c \in S^{p,q}_{\tilde\nu}$ and $1 < p < \infty$, we know that $R_g c$ can be written as the iterated series (16). Write the partial sums of the outer series as

$$S_{K,\infty}c \;=\; \sum_{|k|\le K}\Biggl(\sum_{n\in\mathbb{Z}^d} c_{kn}\, e^{2\pi i\beta n\cdot x}\Biggr) T_{\alpha k}g \;=\; \sum_{|k|\le K} m_k \cdot T_{\alpha k}g.$$

Given $K \ge K_0$ and $N \ge N_0$, write

$$R_g c - S_{K,N} c \;=\; (R_g c - S_{K_0,\infty} c) + (S_{K_0,\infty} c - S_{K_0,N} c) + (S_{K_0,N} c - S_{K,N} c). \tag{28}$$

We will calculate the $W(L^p, L^q_\nu)$ norm of each of these terms separately.

For the first term, define a sequence r by $r_{kn} = c_{kn}$ for $|k| \le K_0$ and $n \in \mathbb{Z}^d$, and $r_{kn} = 0$ otherwise. Then $S_{K_0,\infty} c = R_g r$, and R_g is a bounded mapping of $S^{p,q}_{\tilde\nu} \to W(L^p, L^q_\nu)$, so using (26) we have

$$\begin{aligned} \|R_g c - S_{K_0,\infty} c\|_{W(L^p,L^q_\nu)} &= \|R_g(c-r)\|_{W(L^p,L^q_\nu)} \\ &\le \|R_g\|\,\|c-r\|_{S^{p,q}_{\tilde\nu}} \\ &= \|R_g\| \left(\sum_{|k|>K_0} \|m_k\|^q_{p,Q_{1/\beta}}\, \tilde\nu(k)^q \right)^{1/q} \\ &\le \|R_g\|\varepsilon. \end{aligned} \tag{29}$$

For the second term, define $s_{kn} = c_{kn}$ for $|k| \le K_0$ and $|n| \le N$, and $s_{kn} = 0$ otherwise. Then $S_{K_0,N} c = R_g s$, so using (27), we have

$$\begin{aligned} &\|S_{K_0,\infty} c - S_{K_0,N} c\|_{W(L^p,L^q_\nu)} \\ &\le \|R_g\|\,\|r-s\|_{S^{p,q}_{\tilde\nu}} \\ &= \|R_g\| \left(\sum_{|k|\le K_0} \|m_k - S_N m_k\|^q_{p,Q_{1/\beta}}\, \tilde\nu(k)^q \right)^{1/q} \\ &\le \|R_g\|\varepsilon. \end{aligned} \tag{30}$$

For the third term, define $t_{kn} = c_{kn}$ for $|k| \le K$ and $|n| \le N$, and $t_{kn} = 0$ otherwise. Then $S_{K_0,N} c = R_g t$, so using (25) and (26), we have

$$\begin{aligned} \|S_{K_0,N} c - S_{K,N} c\|_{W(L^p,L^q_\nu)} &\le \|R_g\|\,\|s-t\|_{S^{p,q}_{\tilde\nu}} \\ &= \|R_g\| \left(\sum_{K_0<|k|\le K} \|S_N m_k\|^q_{p,Q_{1/\beta}}\, \tilde\nu(k)^q \right)^{1/q} \\ &\le C_1 \|R_g\| \left(\sum_{K_0<|k|\le K} \|m_k\|^q_{p,Q_{1/\beta}}\, \tilde\nu(k)^q \right)^{1/q} \end{aligned}$$

$$\|S_{K_0,N}c - S_{K,N}c\|_{W(L^p,L^q_\nu)} \le C_1 \|R_g\| \varepsilon. \tag{31}$$

Applying (29)–(31) to (28), we see that $\|R_g c - S_{K,N}c\|_{W(L^p,L^q_\nu)} \le (2 + C_1)\,\|R_g\|\,\varepsilon$, which completes the proof. ☐

3.5 Necessary conditions on the window

In this section we prove a partial converse to Theorem 3.2.6. In particular, Theorem 3.2.6 implies that if $g \in W(L^\infty, L^1_\omega)$, then C_g is bounded on each $W(L^p, L^q_\nu)$. In the converse direction, if g is a measurable function and $1 \le p, q \le \infty$ are given, then in order for C_g to be well-defined on $W(L^p, L^q_\nu)$, we must at least have $C_g f(0,0) = \langle f, g\rangle = \int f\bar{g}$ defined for each $f \in W(L^p, L^q_\nu)$. Hence $f\bar{g} \in L^1$ for all such f, so we immediately have that g must lie in the Köthe dual of $W(L^p, L^q_\nu)$, which is $W(L^{p'}, L^{q'}_{1/\nu})$.

For the unweighted case, we obtain the following further necessary condition in order that C_g be bounded on $W(L^p, L^\infty)$. For the case $p = 2$, this result was obtained by Balan in [2] and published in [4], [3].

Theorem 3.5.1. *Let α, $\beta > 0$ and $1 < p < \infty$ be given. If $g \in W(L^{p'}, L^1)$ and C_g is a bounded map from $W(L^p, L^\infty)$ to $S^{p,\infty}$, then $g \in W(L^\infty, L^p)$.*

Proof. We assume that $g \in W(L^{p'}, L^1)$ is such that C_g is a bounded map from $W(L^p, L^\infty)$ to $S^{p,\infty}$, where $1 < p < \infty$, and we wish to show that $g \in W(L^\infty, L^p)$. Let us show first that $g \in L^\infty$. If not, then given any $D > 0$ there would exist a set J contained in some cube $\frac{\ell}{\beta} + Q_{1/\beta}$ and with positive measure such that $|g(x)| > D$ on J.

Set $f = \frac{1}{|J|^{1/p}}\, e^{i \arg g}\, \chi_J$. Using the equivalent norm for $W(L^p, L^\infty)$ obtained by replacing α in (7) by $1/\beta$, we have that $\|f\|_{W(L^p,L^\infty)} \le 1$. By hypothesis, $C_g f \in S^{p,\infty}$, so there exist $1/\beta$-periodic functions m_k such that $\hat{m}_k(n) = C_g f(k,n)$. Since

$f \cdot T_{\alpha k}\bar{g} \in L^1$, it is easy to see that m_k is given by (17). In particular, considering $k = 0$ we have

$$\begin{aligned} \|m_0\|_{p,Q_{1/\beta}}^p &= \beta^{-pd} \int_{Q_{1/\beta}} \Big| \sum_{n\in\mathbb{Z}^d} f(x - \tfrac{n}{\beta})\, \bar{g}(x - \tfrac{n}{\beta}) \Big|^p \, dx \\ &= \frac{\beta^{-pd}}{|J|} \int_{\frac{\ell}{\beta}+Q_{1/\beta}} \chi_J(x)\, |g(x)|^p \, dx \\ &\geq \beta^{-pd} D^p. \end{aligned}$$

Hence

$$\begin{aligned} D &\leq \beta^d \sup_{k\in\mathbb{Z}^d} \|m_k\|_{p,Q_{1/\beta}} \\ &= \beta^d \, \|C_g f\|_{S^{p,\infty}} \\ &\leq \beta^d \, \|C_g\| \, \|f\|_{W(L^p,L^\infty)} \leq \beta^d \, \|C_g\|. \end{aligned}$$

But since D is arbitrary, this contradicts the fact that C_g is a bounded mapping. Hence g must be in L^∞.

Now we show that $g \in W(L^\infty, L^p)$. Fix $\varepsilon > 0$, and for each $n \in \mathbb{Z}^d$ define

$$J_n \;=\; \big\{ x \in \tfrac{n}{\beta} + Q_{1/\beta} : |g(x)| \geq \tfrac{1}{2}\|g\|_{\infty, \frac{n}{\beta}+Q_{1/\beta}} \big\}.$$

Then set $J_n' = J_n$ if $|J_n| \leq \varepsilon$, otherwise let J_n' be a subset of J_n of measure ε. Let

$$N_\varepsilon \;=\; \sup\{ N \in \mathbb{N} : |J_n'| \geq \tfrac{\varepsilon}{2} \text{ for all } |n| \leq N \}.$$

Note that $N_\varepsilon \to \infty$ as $\varepsilon \to 0$ (and may even be ∞ for some ε). Define $f = e^{i \arg g} \sum_{|n|\leq N_\varepsilon} \chi_{J_n'}$, and note that $\|f\|_{W(L^p,L^\infty)} \leq \varepsilon^{1/p}$. Therefore $C_g f \in S^{p,\infty}$, and letting m_k be defined by (17), we have

$$\begin{aligned} \|m_0\|_{p,Q_{1/\beta}}^p &= \beta^{-pd} \sum_{|n|\le N_\varepsilon} \int_{\frac{n}{\beta}+Q_{1/\beta}} |g(x)|^p \, \chi_{J_n'}(x)\, dx \\ &\ge \beta^{-pd} \sum_{|n|\le N_\varepsilon} \left(\frac{\|g \cdot T_{\frac{n}{\beta}} \chi_{Q_{1/\beta}}\|_\infty}{2} \right)^p |J_n'| \\ &\ge \beta^{-pd} 2^{-p-1} \varepsilon \sum_{|n|\le N_\varepsilon} \|g \cdot T_{\frac{n}{\beta}} \chi_{Q_{1/\beta}}\|_\infty^p. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{|n|\le N_\varepsilon} \|g \cdot T_{\frac{n}{\beta}} \chi_{Q_{1/\beta}}\|_\infty^p &\le \frac{\beta^{pd} 2^{p+1}}{\varepsilon} \sup_{k\in\mathbb{Z}^d} \|m_k\|_{p,Q_{1/\beta}}^p \\ &= \frac{\beta^{pd} 2^{p+1}}{\varepsilon} \|C_g f\|_{S^{p,\infty}}^p \\ &\le \frac{\beta^{pd} 2^{p+1}}{\varepsilon} \|C_g\|^p \, \|f\|_{W(L^p,L^\infty)}^p \\ &\le \beta^{pd} 2^{p+1} \, \|C_g\|^p. \end{aligned}$$

Since $N_\varepsilon \to \infty$ as $\varepsilon \to 0$, this implies that $g \in W(L^\infty, L^p)$. □

Remark 3.5.2. As noted above, the hypothesis $g \in W(L^{p'}, L^1)$ is not a limitation on the generality of the result, as it is necessary in order that C_g can even be defined. Furthermore, if $1 < p < \infty$ then $W(L^\infty, L^p)$ is not contained in $W(L^{p'}, L^1)$ nor conversely, so Theorem 3.5.1 is not a trivial consequence of embeddings of amalgam spaces. The result is also true if $p = 1$, but in this case $W(L^\infty, L^p) = W(L^{p'}, L^1)$ and there is no new information gained.

We now show that, with a mild hypothesis, we obtain a necessary condition on the analysis window. This hypothesis is formulated in terms of the following condition; we refer to [14], [2] for examples.

Definition 3.5.3. A function $f\colon \mathbb{R}^d \to \mathbb{C}$ has persistency length a if there exists a $\delta > 0$ and a compact set K congruent to Q_a mod a such that $|f(x)| \geq \delta$ for every $x \in K$.

Theorem 3.5.4. *Let α, $\beta > 0$ and $1 \leq p < \infty$ be given. Let g, γ be measurable functions on $\mathbb{R}^d$. Suppose the following:*

a. *for each $f \in W(L^p, L^\infty)$, the series $\sum_{k,n} \langle f, M_{\beta n} T_{\alpha k} g\rangle\, M_{\beta n} T_{\alpha k}\gamma$ converges unconditionally in L^p_{loc},*

b. *the frame operators $S_g = R_g C_g$ and $S_\gamma = R_\gamma C_\gamma$ are bounded mappings of $W(L^p, L^\infty)$ onto itself,*

c. *γ has persistency length $1/\beta$.*

Then $g \in W(L^\infty, L^p)$.

Proof. Let $F \subset \mathbb{R}^d$ be compact. Then by hypothesis, $f \mapsto \sum_{k,n} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma$ is a bounded mapping from $W(L^p, L^\infty)$ into L^p_{Loc}, and the series converges unconditionally in L^p_{loc}. We first show that $f \mapsto S_k f = \sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma$ is uniformly bounded on $W(L^p, L^\infty)$, with respect to $k \in \mathbb{Z}^d$.

Fix $k \in \mathbb{Z}^d$ and any $f \in W(L^p, L^\infty)$. Then the sequence

$$\sum_{|n|\leq N} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma$$

converges in $L^p(F)$ as $N \to \infty$, hence is a bounded sequence in $L^p(F)$. Moreover, because the operators

$$S_{N,k}(\cdot) := \sum_{|n|\leq N} \langle \cdot, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma,$$

are bounded, we conclude from the Uniform Boundedness Principle that for each fixed k the sequence of operators $\{S_{N,k}\}_{N\geq 0}$, are uniformly bounded in N, i.e., for each k

there is a constant $C_k > 0$ such that

$$\|S_{N,k}\|_{W(L^p,L^\infty)\to L^p(F)} \leq C_k \forall N.$$

Next, given $\epsilon > 0$ and $f \in W(L^p, L^\infty)$ with $\|f\|_{W(L^p,L^\infty)} = 1$, there exists $N_0 > 0$ such that $\| \sum_{|n|\geq N_0} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma\|_{p,F} \leq \epsilon$. Thus,

$$\begin{aligned}\|S_k(f)\|_{p,F} &\leq \left\| \sum_{|n|\geq N_0} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma \right\|_{p,F} + \\ &\qquad \left\| \sum_{|n|\leq N_0} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma \right\|_{p,F} \\ &\leq C_k + \epsilon.\end{aligned}$$

Letting $\epsilon \to 0$, we conclude that $\|S_k(f)\|_{p,F} \leq C_k$. It then follows that S_k is a bounded operator from $W(L^p, L^\infty)$ into $L^p(F)$. Now because $\sum_{k\in\mathbb{Z}^d} S_k(f)$ converges in $L^p(F)$, we know that $S_k(f)$ is bounded independently of k, so by the Uniform Boundedness principle, we conclude that $\|S_k\|_{W(L^p,L^\infty)\to L^p(F)} \leq C(F)$ for all $k \in \mathbb{Z}^d$. Moreover, it is easy to see that

$$\|S_k\|_{W(L^p,L^\infty)\to L^p(F+\alpha)} = \|S_{k-1}\|_{W(L^p,L^\infty)\to L^p(F)} \leq C(F).$$

Thus if we let $F = Q_\alpha$ then $\|S_k(f)\|_{p,F} \leq C(F)\|f\|_{W(L^p,L^\infty)}$. On the other hand, taking F and $\delta > 0$ as in the definition of persistency, for any $k \in \mathbb{Z}^d$ we have

$$\begin{aligned}\left\| \sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma \right\|_{p,F+\alpha k} &= \|S_k(f)\|_{p,F+\alpha k} \\ &= \left\| \gamma(\cdot) \sum_{n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle e^{2\pi i\beta(\cdot+\alpha k)} \right\|_{p,F} \\ &\geq \delta\beta^d \|m_k\|_{p,Q_{1/\beta}}.\end{aligned}$$

Therefore,

$$\|m_k\|_{p,Q_{1/\beta}} \leq \delta^{-1}\beta^d \|S_k(f)\|_{p,F+\alpha k} \leq C\delta^{-1}\beta^d \|f\|_{W(L^p,L^\infty)} \quad \forall k \in \mathbb{Z}^d.$$

Hence,

$$\|C_g f\|_{S^{p,\infty}} \; = \; \sup_k \|m_k\|_{p,Q_{1/\beta}} \leq C\,\delta^{-1}\,\beta^d\,\|f\|_{W(L^p,L^\infty)}.$$

Thus, we conclude by Theorem 3.5.1 that $g \in W(L^p, L^\infty)$, which concludes the proof.

$\square$

CHAPTER IV

EMBEDDINGS OF BESOV, TRIEBEL-LIZORKIN SPACES INTO MODULATION SPACES

The apparently simple definition of the modulation spaces (see Chapter 2) hides the practical problem of how to decide whether or not a distribution belongs to a given modulation space. In principle one has to estimate the $L^{p,q}_\nu$ norm of the STFT, which can be a non-trivial task. Therefore it is important to understand the relationship between time-frequency content and other properties of distributions, e.g., smoothness properties. Such relationships may appear in the form of embeddings of certain spaces that measure smoothness and/or decay into modulation spaces. For example, Gröchenig in [40], Galperin and Gröchenig in [32], and Hogan and Lakey in [47] derived sufficient conditions for membership in the modulation space M^1 from certain uncertainty principles related to the STFT. Another interesting example appears in [45], where Heil, Ramanathan and Topiwala obtained an embedding that is particularly important in relation to pseudodifferential operator theory.

In the present chapter we prove sufficient conditions for a tempered distribution to belong to certain (unweighted) modulation spaces by proving some embeddings of classical Banach spaces such as the Besov, Triebel-Lizorkin, or Sobolev spaces into the modulation spaces. As corollaries, we obtain some embeddings which generalize the embedding from [45] mentioned above, and, moreover, we will give an easy sufficient condition for membership of a distribution in M^1 in the special case of dimension $d = 1$.

Other embeddings results between modulation spaces and Besov spaces were arrived at independently and by different techniques by P. Gröbner [37], and J. Toft [56].

4.1 The Besov and Triebel-Lizorkin Spaces

Let $\psi \in \mathcal{S}$ be a function such that

$$\begin{cases} 0 \leq \psi(x) \leq 1, \\ \quad \psi(x) = 1, \quad \text{if} \quad |x| \leq 1, \\ \quad \psi(x) = 0, \quad \text{if} \quad |x| \geq 3/2. \end{cases}$$

Define

$$\begin{cases} \phi_0(x) = \psi(x), \\ \phi_1(x) = \psi(\frac{x}{2}) - \psi(x), \\ \phi_k(x) = \phi_1(2^{-k+1}x), \ k = 2, 3, ... \end{cases}$$

Then $\{\phi_k\}_{k=0}^{\infty}$ is a partition of unity, and satisfies

$$supp(\phi_k) \subset \{x \in \mathbb{R}^d : 2^{k-1} \leq |x| \leq 3 \cdot 2^{k-1}\}.$$

Definition 4.1.1. Let $s \in \mathbb{R}$, $1 \leq q \leq \infty$, and $f \in \mathcal{S}'$.

(i) For $1 \leq p < \infty$ the Triebel-Lizorkin space $F^s_{p,q}$ is defined by:

$$f \in F^s_{p,q} \iff \|f\|_{F^s_{p,q}} = \left(\int_{\mathbb{R}^d} \left(\sum_{k=0}^{\infty} 2^{skq} \, |\mathcal{F}^{-1}(\phi_k \hat{f})(x)|^q\right)^{p/q} dx\right)^{1/p} < \infty. \tag{32}$$

(ii) For $1 \leq p \leq \infty$, the Besov space $B^s_{p,q}$ is defined by:

$$f \in B^s_{p,q} \iff \|f\|_{B^s_{p,q}} = \left(\sum_{k=0}^{\infty} 2^{skq} \left(\int_{\mathbb{R}^d} |\mathcal{F}^{-1}(\phi_k \hat{f})(x)|^p \, dx\right)^{q/p}\right) < \infty. \tag{33}$$

(iii) For $p = \infty$, the Triebel-Lizorkin space $F^s_{\infty,q}$ is defined by:

$$f \in F^s_{\infty,q} \iff \exists \{f_k\}_{k=0}^{\infty}, \quad f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\phi_k \hat{f}_k), \quad \sup_{\mathbb{R}^d} \left(\sum_{k=0}^{\infty} 2^{ksq} \, |f_k(x)|^q\right)^{1/q} < \infty, \tag{34}$$

with norm

$$\|f\|_{F^s_{\infty,q}} = \inf\left(\sup_{\mathbb{R}^d}\left(\sum_{k=0}^{\infty} 2^{ksq}|f_k(x)|^q\right)^{1/q}\right),$$

the infimum being taken over all admissible representations.

(iv) For $1 \leq p \leq \infty$, the fractional Sobolev space H^s_p is defined by:

$$f \in H^s_p \iff \|f\|_{H^s_p} = \left(\int_{\mathbb{R}^d} |\mathcal{F}^{-1}((1+|x|^2)^{s/2}\hat{f})(x)|^p\, dx\right)^{1/p} < \infty. \tag{35}$$

Remark 4.1.2. The classes of Besov and Triebel-Lizorkin spaces comprise many of the spaces encounter in analysis, e.g., we have the following identifications whose proofs may be found in [57, Sect. 2.3.5].

a. If $1 \leq p = q \leq \infty$ and $s \in \mathbb{R}$, then $B^s_{p,p} = F^s_{p,p}$, this follows from the definition.

b. If $1 < p < \infty$, then $F^0_{p,2} = L^p$.

c. If $1 < p < \infty$ and $s \in \mathbb{R}$, then $F^s_{p,2} = H^s_p$.

Moreover, for $1 \leq p, q \leq \infty$ and $s > 0$ we have that $B^s_{p,q} \subset L^p$, additionally if $p < \infty$ we also have $F^s_{p,q} \subset L^p$. We refer to [58, Sect. 2.3.2] for the proof of these last assertions.

More generally, we refer to [57], [58], [53] and [55] for background and information about the Triebel-Lizorkin, Besov, and Sobolev spaces.

Because the Besov and the Tribel-Lizorkin spaces have been rediscovered (under different names) by various authors, they have a number of equivalent definitions. We collect here some of those results that will be needed in the sequel: Propositions 4.1.3 and 4.1.4 give equivalent definitions of $F^s_{p,q}$ and $B^s_{p,q}$, respectively, while Proposition 4.1.5 is a result on interpolation of Besov spaces.

The following result is proved in [57, Proposition 1, 2.3.4].

Proposition 4.1.3. *Let $s \in \mathbb{R}$, $1 < p \le \infty$, and $1 < q \le \infty$. If $f \in \mathcal{S}'$ then $f \in F^s_{p,q}$ if and only if*

$$\exists \{f_k\}_{k=0}^{\infty} \subset L^p,\ f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\phi_k \hat{f}_k), \quad \left(\int_{\mathbb{R}^d} \left(\sum_{k=0}^{\infty} 2^{ksq} |f_k(x)|^q\right)^{p/q} dx\right)^{1/p} < \infty. \tag{36}$$

Furthermore,

$$\inf \left(\int_{\mathbb{R}^d} \left(\sum_{k=0}^{\infty} 2^{ksq} |f_k(x)|^q\right)^{p/q} dx\right)^{1/p}$$

is an equivalent norm on $F^s_{p,q}$, where the infimum is taken over all admissible representations of f.

See [53, Theorem 2, 2.3.2] for a proof of the following result.

Proposition 4.1.4. *Let $1 \le p, q \le \infty$ and $s > 0$. If $f \in \mathcal{S}'$ then $f \in B^s_{p,q}$ if and only if*

$$\exists \quad \{b_k\}_{=0}^{\infty} \subset L^p, \quad f = \sum_{k=0}^{\infty} b_k, \ \left(\sum_{k=0}^{\infty} 2^{kqs} \left(\int_{\mathbb{R}^d} |b_k(x)|^p \, dx\right)^{q/p}\right)^{1/q} < \infty. \tag{37}$$

Furthermore,

$$\inf \left(\sum_{k=0}^{\infty} 2^{kqs} \left(\int_{\mathbb{R}^d} |b_k(x)|^p \, dx\right)^{q/p}\right)^{1/q}$$

is an equivalent norm on $B^s_{p,q}$, where the infimum is taken over all admissible representations of f.

See [57, Theorem 2.4.7] for a proof of the following result about complex interpolation of Besov spaces.

Proposition 4.1.5. *Let $s_0, s_1 \in \mathbb{R}$, $1 \le p_0, q_0, p_1, q_1 \le \infty$, and $0 < \theta < 1$. If $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ then*

$$\left(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}\right)_{[\theta]} = B^s_{p,q}. \tag{38}$$

The next proposition collects some of the computations involved in the proofs of our results. Part (a) computes the STFT of a Gaussian with respect to a dilated

Gaussian. The result is essentially the product of two Gaussians (one in time and the other in frequency). Part (b) shows that the inverse Fourier transform of the Bessel potential

$$m_{-s}(x) = (1 + |x|^2)^{-s/2}$$

is in M^1 for $s > d$. Because M^1 is invariant under Fourier transforms we then conclude that the Bessel potential m_{-s} itself is in M^1 for $s > d$.

Proposition 4.1.6. *Define* $g(x) = e^{-\pi x^2}$ *and* $g_a(x) = e^{-\frac{\pi x^2}{a}}$. *Let*

$$G_s(x) = \frac{1}{(4\pi)^{s/2}} \frac{1}{\Gamma(s/2)} \int_0^\infty t^{\frac{-d+s}{2}} e^{-(\frac{\pi x^2}{t} + \frac{t}{4\pi})} \frac{dt}{t}$$

for $s, a > 0$ *and* $x \in \mathbb{R}^d$, *and where* Γ *refers to the Gamma-function. Then the following hold.*

(a) $V_{g_a} g(x, \omega) = \left(\frac{a}{a+1}\right)^{d/2} e^{2\pi i \frac{x \cdot \omega}{a+1}} \, g_{a+1}(x) \, g_{\frac{a+1}{a}}(\omega)$.

(b) $V_g \check{m}_{-s} = (2\pi)^{d/2} \, V_g(D_{\frac{1}{2\pi}} G_s) \in L^1$ *for* $s > d$, *where* D_a *is the unitary dilation operator defined by* $D_a g(x) = |a|^{-d/2} g(x/a)$.

(c) $m_{-s} \in M^1$ *for* $s > d$.

Proof. (a) First note that from Lemma 2.2.2 the operator $D_a f(t) = |a|^{-d/2} \, f(t/a)$ where $a > 0$ is unitary on L^2, and $\widehat{D_a f} = D_{1/a} \hat{f}$. Now for $x, \omega \in \mathbb{R}^d$ we have:

$$\begin{aligned}
V_{g_a} g(x, \omega) &= \int_{\mathbb{R}^d} e^{-\pi \frac{t^2}{a}} \, e^{-2\pi i t \cdot \omega} \, e^{-\pi (t-x)^2} \, dt \\
&= \int_{\mathbb{R}^d} e^{-\frac{\pi}{a}\left((a+1)t^2 - 2a t \cdot x + a x^2\right)} \, e^{-2\pi i t \cdot \omega} \\
&= \int_{\mathbb{R}^d} e^{-\pi \frac{a+1}{a}\left((t - \frac{a}{a+1}x)^2 + \frac{a}{a+1}x^2\right)} \, e^{-2\pi t \cdot \omega} \, dt \\
&= e^{-\frac{\pi}{a+1}x^2} \int_{\mathbb{R}^d} e^{-\pi \frac{a+1}{a}(t - \frac{a}{a+1}x)^2} \, e^{-2\pi t \cdot \omega} \, dt \\
&= e^{-\frac{\pi}{a+1}x^2} \, (T_{\frac{a}{a+1}x} \, g_{\frac{a}{a+1}})^\wedge(\omega)
\end{aligned}$$

$$= g_{a+1}(x)\, M_{-\frac{a}{a+1}x}\, \widehat{g_{\frac{a}{a+1}}}(\omega)$$

$$= \big(\frac{a}{a+1}\big)^{d/2}\, g_{a+1}(x)\, M_{-\frac{a}{a+1}x}\, g_{\frac{a+1}{a}}(\omega)$$

$$= \big(\frac{a}{a+1}\big)^{d/2}\, e^{-2\pi i \frac{a}{a+1} x\cdot\omega}\, g_{a+1}(x)\, g_{\frac{a+1}{a}}(\omega),$$

where we have used the fact that the Fourier transform of the Gaussian $g(x) = e^{-\pi x^2}$ is itself, i.e., $\hat{g} = g$.

(b) For $s > 0$ it is shown in [55, Proposition 3.1.2] that

$$\hat{G}_s(\omega) = (1 + 4\pi^2|x|^2)^{-s/2}.$$

Notice for future references that $G_s \in L^1$, see [55, Proposition 3.1.2]. Thus,

$$\begin{aligned} m_{-s}(x) &= (1 + |x|^2)^{-s/2} \\ &= \hat{G}_s(x/2\pi) \\ &= (2\pi)^{d/2}\, D_{2\pi}\hat{G}_s(x) \\ &= (2\pi)^{d/2}\, \widehat{D_{1/2\pi} G_s}(x). \end{aligned}$$

Consequently, we have that $\check{m}_{-s}(x) = (2\pi)^{d/2}\, (D_{\frac{1}{2\pi}} G_s)(x)$. Therefore:

$$\begin{aligned} V_g\check{m}_{-s}(x,\omega) &= (2\pi)^{d/2}\, V_g(D_{\frac{1}{2\pi}} G_s)(x,\omega) \\ &= (2\pi)^{d/2} \int_{\mathbb{R}^d} D_{\frac{1}{2\pi}} G_s(t)\, e^{-2\pi i t\cdot\omega}\, \overline{g}(t-x)\, dt, \\ &= (2\pi)^{d/2} \big\langle D_{\frac{1}{2\pi}} G_s,\ M_\omega T_x g \big\rangle \\ &= (2\pi)^{d/2} \big\langle G_s,\ D_{2\pi} M_\omega T_x g \big\rangle \\ &= \big\langle G_s,\ M_{\frac{\omega}{2\pi}} T_{2\pi x} g_{2\pi} \big\rangle \\ &= \frac{1}{(4\pi)^{s/2}\,\Gamma(s/2)} \int_{\mathbb{R}^d} \int_0^\infty u^{\frac{s-d}{2}}\, e^{-(\frac{\pi t^2}{u} + \frac{u}{4\pi})}\, e^{-2\pi i t\cdot\omega/2\pi} \times \\ &\qquad\qquad e^{-\pi(t-2\pi x)^2/2\pi}\, \frac{du}{u}\, dt \end{aligned}$$

$$= \frac{1}{(4\pi)^{s/2}\,\Gamma(s/2)} \int_0^\infty e^{-u/4\pi}\, u^{\frac{s-d}{2}-1} \int_{\mathbb{R}^d} e^{-\frac{\pi t^2}{u}} \times$$

$$e^{-2\pi i t\cdot\omega/2\pi}\, e^{-\pi\frac{(t-2\pi x)^2}{2\pi}}\, dt\, du$$

$$= \frac{1}{(4\pi)^{s/2}\,\Gamma(s/2)} \int_0^\infty e^{-u/4\pi}\, u^{\frac{s}{2}-1} \int_{\mathbb{R}^d} e^{-\pi t^2} \times$$

$$e^{-2\pi i t\cdot\omega\frac{\sqrt{u}}{2\pi}}\, e^{-\pi u\frac{(t-\frac{2\pi x}{\sqrt{u}})^2}{2\pi}}\, dt\, du$$

$$= \frac{1}{(4\pi)^{s/2}\,\Gamma(s/2)} \int_0^\infty e^{-u/4\pi}\, u^{\frac{s}{2}-1}\, V_{g_{\frac{2\pi}{u}}} g(\frac{2\pi x}{\sqrt{u}}, \frac{\sqrt{u}\omega}{2\pi})\, du.$$

The last equality follows from some changes of variable and by using part a. Therefore, we have that:

$$V_g\check{m}_{-s}(x,\omega) = \frac{\pi^{(d-s)/2}}{2^{s-d/2}\,\Gamma(s/2)} \int_0^\infty e^{-u/4\pi}\, e^{-\pi\frac{4\pi^2 x^2}{u+2\pi}}\, e^{-\pi\frac{u}{2\pi(2\pi+u)}\,\omega^2} \times$$

$$u^{\frac{s}{2}-1}\, \frac{1}{(2\pi+u)^{d/2}}\, du,$$

where we have used Lemma 2.2.2 to obtain the last equation. By changing the variables and using the fact that $\int_{\mathbb{R}^d} e^{-\pi x^2}\, dx = 1$, we have:

$$\|V_g\check{m}_{-s}\|_{L^1} = \iint_{\mathbb{R}^{2d}} |V_g\check{m}_{-s}(x,\omega)|\, dx\, d\omega$$

$$\leq \frac{\pi^{(d-s)/2}}{2^{s-d/2}} \int_0^\infty e^{-u/4\pi}\, u^{\frac{s}{2}-1}\, \frac{1}{(2\pi+u)^{d/2}} \times$$

$$\int_{\mathbb{R}^d} e^{-\pi\frac{4\pi^2 x^2}{2\pi+u}} \int_{\mathbb{R}^d} e^{-\pi\frac{u}{2\pi(2\pi+u)}\omega^2}\, d\omega\, dx\, du$$

$$= \frac{\pi^{(d-s)/2}}{2^{s-d/2}} \int_0^\infty e^{-u/4\pi}\, u^{\frac{-d+s}{2}-1}\, (u+2\pi)^{d/2}\, du, \qquad (39)$$

and the last expression is finite if $s > d$.

(c) Follows from (b) and the comments above. □

4.2 Embedding of Besov, Triebel-Lizorkin spaces into modulation spaces

There are several embeddings between the Besov or Triebel-Lizorkin and modulation spaces that can easily be derived. Some of these embeddings are summarized in the following result.

Proposition 4.2.1. (a) $B^s_{p,q} \subset L^p \subset M^{p,p'}$ *when* $s > 0$, $1 \leq p \leq 2$, *and* $1 \leq q \leq \infty$.
(b) $B^s_{p,q} \subset L^p \subset M^p$ *when* $s > 0$, $2 \leq p \leq \infty$, *and* $1 \leq q \leq \infty$.

Proof. (a) The first of these embeddings was mentioned in Remark 4.1.2, and its proof can be found in [58, Remark 3, Sect. 2.3.2]. To prove the second one, let $f \in L^p$, and let $g \in \mathcal{S}$. Then for $x, \omega \in \mathbb{R}^d$, $V_g f(x,\omega) = \widehat{f \cdot T_x \overline{g}}(\omega)$. Note that the $f \cdot T_x g \in L^p$ since $f \in L^p$ and $g \in \mathcal{S}$. Moreover, since $1 \leq p \leq 2 \leq p'$, we have that $p'/p \geq 1$, and thus using Hausdorff-Young's inequality and Minkowski's inequality (for integrals) we have:

$$\begin{aligned}
\|V_g f\|^p_{L^{p,p'}} &= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \, dx\right)^{p'/p} d\omega\right)^{p/p'} \\
&\leq \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^{p'} d\omega\right)^{p/p'} dx \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |\widehat{f \cdot T_x \overline{g}}(\omega)|^{p'} d\omega\right)^{p/p'} dx \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f \cdot T_x \overline{g}(t)|^p dt \, dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(t)|^p \, |g(t-x)|^p \, dt \, dx \\
&= \|f\|^p_{L^p} \, \|g\|^p_{L^p}.
\end{aligned}$$

Hence,

$$\|V_g f\|_{L^{p,p'}} = \|f\|_{M^{p,p'}} \leq \|f\|_{L^p} \, \|g\|_p,$$

which concludes the proof the second inclusion.

(b) We now prove the second of these inclusions. Let $f \in L^p$, since $2 \le p \le \infty$, we have that $1 \le p' \le 2$. Moreover, choosing $g \in \mathcal{S}$ we see using Hölder's inequality with $p/p' \ge 1$, that $f \cdot T_x g^* \in L^{p'}$ for almost all $x \in \mathbb{R}^d$. By using Hausdorff-Young's inequality as well as Young's inequality we have that

$$\begin{aligned}
\|V_g f\|_{L^p} &= \left(\iint_{\mathbb{R}^{2d}} |V_g f(x,\omega)|^p \, dx \, d\omega \right)^{1/p} \\
&= \left(\iint_{\mathbb{R}^{2d}} |\widehat{f \cdot T_x \overline{g}}(\omega)|^p \, d\omega \, dx \right)^{1/p} \\
&\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |(f \cdot T_x \overline{g})(t)|^{p'} \, dt \right)^{p/p'} , dx \\
&= \left(\int_{\mathbb{R}^d} \big(\int_{\mathbb{R}^d} |f(t)|^{p'} \, |g^*(x-t)|^{p'} \, dt \big)^{p/p'} \, dx \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^d} \big(|f|^{p'} * |g^*|^{p'}(x) \big)^{p/p'} \, dx \right)^{1/p} \\
&= \big\| |f|^{p'} * |g^*|^{p'} \big\|_{L^{p/p'}}^{1/p'} .
\end{aligned}$$

Now notice that $p \ge p'$, and that $f \in L^p \iff |f|^{p'} \in L^{p/p'}$. Thus, applying Young's inequality with parameter p/p' we have that

$$\begin{aligned}
\|V_g f\|_{L^p}^{p'} &\le \big\| |f|^{p'} * |g^*|^{p'} \big\|_{L^{p/p'}} \\
&\le \| |f|^{p'} \|_{L^{p/p'}} \, \big\| |g^*|^{p'} \big\|_{L^1} \\
&= \|f\|_{L^p}^{p'} \, \|g\|_{L^{p'}}^{p'} .
\end{aligned}$$

Consequently,

$$\|f\|_{M^{p,p'}} \le \|g\|_{L^{p'}} \, \|f\|_{L^p},$$

which concludes the proof in this case. □

Other more subtle embeddings of classical spaces into modulation spaces were obtained as byproducts of other results. For example, Gröchenig in [40] and Hogan and Lakey in [47] derived sufficient conditions for membership in the modulation space M^1 from certain uncertainty principles related to the STFT. Precisely, it was shown that $L^p_a \cap \mathcal{F}L^q_b \subset M^1$ under appropriate conditions on p, q and the weight parameters a, b, where L^p_a is a weighted L^p space (with weight $(1+|x|)^p$), and $\mathcal{F}L^q_b$ is the image of L^q_b under the Fourier transform. Somewhat more general embeddings involving weighted L^p spaces were given by Galperin and Gröchenig in [32]. Another interesting example appears in [45]. There, Heil, Ramanathan and Topiwala proved, in our notation, that $\mathcal{C}^s(\mathbb{R}^{2d}) \subset M^{\infty,1}(\mathbb{R}^{2d})$ for $s > 2d$, while working on a time-frequency approach to pseudodifferential operators.

The embeddings we will prove are more difficult, and require an appropriate norm on the Besov or Triebel-Lizorkin space in consideration, along with a correct choice of the form of the STFT. In particular, the following equivalent forms of the STFT (see Proposition 2.3.3) will be useful:

$$\begin{aligned} V_g f(x,\omega) &= \big(f \cdot T_x\overline{g}\big)^{\wedge}(\omega) \\ &= e^{-2\pi i x\omega}\, V_{\hat{g}}\hat{f}(\omega,-x) \\ &= e^{-2\pi i x\omega}\mathcal{F}^{-1}(\hat{f}\cdot T_\omega\overline{\hat{g}})(x) \\ &= e^{-2\pi i x\omega}\big(f * (M_\omega\overline{g})(x)\big). \end{aligned} \tag{40}$$

Our first main embedding result involves the Besov spaces and is as follows.

Theorem 4.2.2. *Let $1 \le p \le 2$ and $1 \le q \le \infty$. If $s > d(\frac{2}{p}-1)$ then*

$$B^s_{p,q} \subset M^{p',p}. \tag{41}$$

Proof. Let $f \in B^s_{p,q}$, and use (37) to write $f = \sum b_k$ where $b_k \in L^p$, and

$$supp(\hat{b}_k) \subset \{|x| \le 2^k\}, \qquad \text{and} \qquad \|f\|_{B^s_{p,q}} \sim \inf\big(\sum_{k=0}^{\infty} 2^{ksq}\|b_k\|^q_{L^p}\big)^{1/q},$$

where the infimum is over all possible such representations of f. Given $g \in \mathcal{S}$, we have using (40) that

$$V_g f(x,\omega) = \sum_{k=0}^{\infty} e^{-2\pi i x\omega}\, \mathcal{F}^{-1}(\hat{b}_k \cdot T_\omega \overline{\hat{g}})(x).$$

Hence by the Hausdorff-Young inequality,

$$\begin{aligned} \|V_g f(\cdot,\omega)\|_{L^{p'}} &\le \sum_{k=0}^{\infty} \|\mathcal{F}^{-1}(\hat{b}_k \cdot T_\omega \overline{\hat{g}})\|_{L^{p'}} \\ &\le \sum_{k=0}^{\infty} \|\hat{b}_k \cdot T_\omega \overline{\hat{g}}\|_{L^p}. \end{aligned}$$

Therefore, by Minkowski's inequality,

$$\begin{aligned} \|V_g f\|_{L^{p',p}} &\le \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^d} \|\hat{b}_k \cdot T_\omega \overline{\hat{g}}\|_{L^p}^p \, d\omega \right)^{1/p} \\ &= \|\hat{g}\|_{L^p} \sum_{k=0}^{\infty} \|\hat{b}_k\|_{L^p}. \end{aligned} \tag{42}$$

Now, $\hat{b}_k$ has compact support, which is contained in $E_k = \{|x| \le 2^k\}$. Since $1 \le p \le 2 \le p'$, we have that $p'/p \ge 1$, and using Hölder's inequality and Hausdorff-Young's inequality we obtain the following estimates

$$\begin{aligned} \|\hat{b}_k\|_{L^p}^p &= \|\hat{b}_k\|_{p,E_k}^p \\ &= \int_{E_k} |\hat{b}_k(\omega)|^p \, d\omega \\ &\le |E_k|^{1-\frac{p}{p'}} \|\hat{b}_k\|_{E_k,p'}^p \\ &\le C\, 2^{kd(1-\frac{p}{p'})} \|\hat{b}_k\|_{p'}^p. \end{aligned}$$

where C is the volume of the ball of center 0 and radius 1. Thus,

$$\begin{aligned} \|\hat{b}_k\|_{L^p} &\le C^{1/p}\, 2^{kd(\frac{1}{p}-\frac{p}{p'})} \|\hat{b}_k\|_{p'} \\ &\le C'\, 2^{kd(\frac{1}{p}-\frac{1}{p'})} \|b_k\|_{L^p}. \end{aligned} \tag{43}$$

Substituting (43) into (42) and applying Hölder's inequality yields

$$\begin{aligned}
\|V_g f\|_{L^{p',p}} &\le C' \,\|\hat{g}\|_p \sum_{k=0}^{\infty} 2^{kd(\frac{1}{p}-\frac{1}{p'})} \,\|b_k\|_p \\
&= C' \,\|\hat{g}\|_p \sum_{k=0}^{\infty} 2^{kd(\frac{1}{p}-\frac{1}{p'}-\frac{s}{d})} \, 2^{ks}\|b_k\|_p \\
&\le C' \left(\sum_{k=0}^{\infty} 2^{ksq} \,\|b_k\|_{L^p}^q\right)^{1/q} \left(\sum_{k=0}^{\infty} 2^{kq'd(\frac{1}{p}-\frac{1}{p'}-\frac{s}{d})}\right)^{1/q'} \\
&\le C \,\|f\|_{B^s_{p,q}} \left(\sum_{k=0}^{\infty} 2^{kq'd(\frac{1}{p}-\frac{1}{p'}-\frac{s}{d})}\right)^{1/q'}. \qquad (44)
\end{aligned}$$

The last term in (44) is finite if and only if $s > d(\frac{1}{p} - \frac{1}{p'})$. □

The next result recovers and extends the embedding in [45], and follows by identifying $F^s_{p,p}$ with $B^s_{p,p}$ (see Remark 4.1.2) and by using Proposition 4.1.3 as the appropriate definition of $B^s_{p,p}$. However, it does not include the fact that $B^s_{2,2} \subset L^2 = M^2$ for $s \ge 0$. This last embedding is obtained as corollary by using complex interpolation methods.

Theorem 4.2.3. *Let $1 \le p \le \infty$. If $s > \frac{d}{p'}$ then*

$$B^s_{p,p} \subset M^{p,p'}. \qquad (45)$$

Proof. We divide the proof in two cases.

Case 1: $p = 1$. Let $f \in B^s_{1,1}$, if $f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\phi_k \hat{f})$, then letting $g(x) = e^{-\pi x^2}$, we have

$$V_g f(x,\omega) = \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\hat{f}\phi_k) * \mathcal{F}^{-1}(T_\omega \overline{\hat{g}})(x).$$

Therefore we have the following estimates

$$\|V_g f(\cdot,\omega)\|_{L^1} \le \sum_{k=0}^{\infty} \|\mathcal{F}^{-1}(\phi_k \,\hat{f})\|_1 \,\|\mathcal{F}^{-1}(\hat{\delta} \cdot T_\omega \overline{\hat{g}})\|_1$$

$$\|V_g f\|_{L^{1,\infty}} \le \sum_{k=0}^{\infty} \|\mathcal{F}^{-1}(\phi_k \,\hat{f})\|_1 \sup_{\omega \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}^{-1}(\hat{\delta} \cdot T_\omega \overline{\hat{g}})(x)| \, dx$$

$$\leq \sum_{k=0}^{\infty} \|\mathcal{F}^{-1}(\phi_k\,\hat{f})\|_1 \sup_{\omega\in\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g\delta(x,\omega)|\,dx.$$

However,

$$\sup_{\omega\in\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g\delta(x,\omega)|\,dx = \sup_{\omega\in\mathbb{R}^d} \int_{\mathbb{R}^d} |M_\omega\, T_x\, g(0)|\,dx = \int_{\mathbb{R}^d} |g(x)|\,dx < \infty.$$

Thus,

$$\begin{aligned} \|V_g f\|_{L^{1,\infty}} &\leq \|g\|_1 \sum_{k=0}^{\infty} \|\mathcal{F}^{-1}(\phi_k\,\hat{f})\|_1 \\ &\leq \|g\|_1 \sum_{k=0}^{\infty} 2^{-ks}\,\|\mathcal{F}^{-1}(\phi_k\,\hat{f})\|_1, \end{aligned}$$

for all $s > 0$. Hence, using (33) we have that

$$\|f\|_{M^{1,\infty}} \leq \|g\|_1\,\|f\|_{B^s_{1,1}},$$

which concludes the proof for this case.

Case 2: Assume $1 < p \leq \infty$. By (36) with $p = q$ we obtain an equivalent norm for $F^s_{p,p} = B^s_{p,p}$. Let $f \in B^s_{p,p}$, then $f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\phi_k \hat{f}_k)$. Let $g(x) = e^{-\pi x^2}$. Then, using (40),

$$V_g f(x,\omega) = \sum_{k=0}^{\infty} f_k * \mathcal{F}^{-1}(\phi_k \cdot T_\omega \overline{\hat{g}})(x),$$

so by Young's convolution inequality,

$$\|V_g f(.,\omega)\|_{L^p} \leq \sum_{k=0}^{\infty} \|f_k\|_{L^p}\,\|\mathcal{F}^{-1}(\phi_k \cdot T_\omega \overline{\hat{g}})\|_{L^1}.$$

Hence, by Minkowski's inequality and Hölder's inequality,

$$\begin{aligned} \|V_g f\|_{L^{p,p'}} &\leq \sum_{k=0}^{\infty} \|f_k\|_{L^p} \left(\int_{\mathbb{R}^d} \left(\|\mathcal{F}^{-1}(\phi_k \cdot T_\omega \overline{\hat{g}})\|_{L^1}\right)^{p'} d\omega\right)^{1/p'} \\ &= \sum_{k=0}^{\infty} 2^{sk}\,\|f_k\|_{L^p}\,2^{-sk} \left(\int_{\mathbb{R}^d} \left(\|\mathcal{F}^{-1}(\phi_k \cdot T_\omega \overline{\hat{g}})\|_{L^1}\right)^{p'} d\omega\right)^{1/p'} \\ &\leq \left(\sum_{k=0}^{\infty} 2^{ksp}\,\|f_k\|^p_{L^p}\right)^{1/p} \left(\sum_{k=0}^{\infty} 2^{-ksp'} \int_{\mathbb{R}^d} \|\mathcal{F}^{-1}(\phi_k \cdot T_\omega \overline{\hat{g}})\|^{p'}_{L^1}\,d\omega\right)^{1/p'}, \end{aligned}$$

and therefore

$$\|f\|_{M^{p,p'}} \le \|f\|_{B^s_{p,p}} \left(\sum_{k=0}^{\infty} 2^{-ksp'} \int_{\mathbb{R}^d} \|\mathcal{F}^{-1}(\phi_k \cdot T_\omega \overline{\hat{g}})\|_{L^1}^{p'} \, d\omega \right)^{1/p'}. \tag{46}$$

Now we will estimate the terms in the summation on the right-hand side of (46). Setting $g_k(x) = g(2^{-k+1}x)$, i.e., $g_k = 2^{\frac{k-1}{2}} D_{2^{k-1}} g$, we have:

$$\left(\int_{\mathbb{R}^d} \|\mathcal{F}^{-1}(\phi_k \cdot T_\omega \overline{\hat{g}})\|_{L^1}^{p'} \, d\omega \right)^{1/p'} = \|V_g \check{\phi}_k\|_{L^{1,p'}}.$$

But, using Lemma 2.2.2, we obtain:

$$\begin{aligned}
V_g \check{\phi}_k(x,\omega) &= \left\langle \check{\phi}_k,\ M_\omega T_x g \right\rangle \\
&= \left\langle \phi_k,\ T_\omega M_{-x} \hat{g} \right\rangle \\
&= 2^{(k-1)/2} \left\langle D_{2^{k-1}} \phi_1,\ T_\omega M_{-x} \hat{g} \right\rangle \\
&= 2^{(k-1)/2} \left\langle \phi_1,\ T_{2^{1-k}\omega} M_{-2^{k-1}x} \widehat{D_{2^{k-1}} g} \right\rangle \\
&= 2^{(k-1)/2} \left\langle \check{\phi}_1,\ M_{2^{1-k}\omega} T_{2^{k-1}x} D_{2^{k-1}g} \right\rangle \\
&= \left\langle \check{\phi}_1,\ M_{2^{1-k}\omega} T_{2^{k-1}x} g_k \right\rangle \\
&= V_{g_k} \check{\phi}_1(2^{k-1}x, 2^{1-k}\omega).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\int_{\mathbb{R}^d} \|\mathcal{F}^{-1}(\phi_k \cdot T_\omega \overline{\hat{g}})\|_{L^1}^{p'} \, d\omega \right)^{1/p'} &= \|V_g \check{\phi}_k\|_{L^{1,p'}} \\
&= \|V_{g_k} \check{\phi}_1(2^{k-1}\cdot, 2^{-k+1}\cdot)\|_{L^{1,p'}} \\
&= 2^{d(-k+1)}\, 2^{d(k-1)/p'} \, \|V_{g_k} \check{\phi}_1\|_{L^{1,p'}} \\
&\le C_1\, 2^{d/p}\, 2^{-kd/p} \, \|V_{g_k} g\|_{L^{1,1}} \, \|V_g \check{\phi}_1\|_{L^{1,p'}},
\end{aligned} \tag{47}$$

the last inequality following from the independence of the definition of the modulation space with respect to the window used to compute the STFT (see Proposition 2.3.6).

Using Proposition 4.1.6 with $a = 2^{k-1}$, we have that

$$\|V_{g_k} g\|_{L^1} = \|V_{g_a} g\|_{L^1} = (1 + 2^{2k-2})^{d/2}. \tag{48}$$

Combining (46), (47) and (48) yields:

$$\|f\|_{M^{p,p'}} \leq C_2 \|f\|_{B^s_{p,p}} \left(\sum_{k=0}^{\infty} 2^{-kp'(s-d+d/p)} \right)^{1/p'}. \tag{49}$$

The last term in the right-hand side of (49) is finite if and only if $s > d(1-\frac{1}{p}) = \frac{d}{p'}$. $\square$

Corollary 4.2.4. *If* $2 \leq p \leq \infty$ *and* $s > d(1 - \frac{2}{p})$ *then*

$$B^s_{p,p} \subset M^{p,p'}. \tag{50}$$

Proof. We will prove this part by interpolating between the cases $p = 2, s_0 \geq 0$ and $p = \infty, s_1 > d$. In particular, we trivially have

$$B^{s_0}_{2,2} = F^{s_0}_{2,2} = H^{s_0}_2 \subset L^2 = M^2 \quad \text{for} \quad s_0 \geq 0, \tag{51}$$

and applying Theorem 4.2.3 to $p = \infty$ yields:

$$B^{s_1}_{\infty,\infty} \subset M^{\infty,1} \quad \text{for} \quad s_1 > d. \tag{52}$$

By [57, Remark 4, Sect. 2.4.1] we have that

$$\left(B^{s_0}_{2,2}, B^{s_1}_{\infty,\infty}\right)_{[\theta]} \subset \left(M^{2,2}, M^{\infty,1}\right)_{[\theta]}$$

for appropriate values of s and θ. We now apply the interpolation result of modulation spaces Proposition 2.3.10, with $p_0 = q_0 = 2$ and $p_1 = \infty, q_1 = 1$. Hence $\frac{1}{p} = \frac{1-\theta}{2}$, and $\frac{1}{q} = \frac{1-\theta}{2} - \theta$. Consequently, $\frac{1}{p} + \frac{1}{q} = 1$. It follows from the above referenced proposition that

$$\left(M^{2,2}, M^{\infty,1}\right)_{[\theta]} = M^{p,q} = M^{p,p'}.$$

Similarly, if we apply the interpolation of Besov spaces Proposition 4.1.5 with $s_0 = 0$, $s_1 > d$, $p_0 = q_0 = 2$, and $p_1 = q_1 = \infty$, hence $s = \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{2} = \frac{1}{q}$, i.e., $p = q$. It follows that

$$\left(B^0_{2,2}, B^{s_1}_{\infty,\infty}\right)_{[\theta]} = B^s_{p,p},$$

and moreover, because $0 < \theta = 1 - 2/p < 1$, we have that $p \geq 2$, hence, $s = s_1\theta > d(1 - \frac{2}{p})$, which concludes the proof. □

Our next result yields an embedding of a fractional Sobolev space (or Bessel potential space) into a modulation space. This can also be seen as an embedding of the Triebel-Lizorkin space $F^s_{p,2}$ into a modulation space, since $F^s_{p,2} = H^s_p$ when $1 < p < \infty$ [57, 58, 53].

Theorem 4.2.5. *If $1 \leq p \leq \infty$, then*

$$H^s_p \subset M^{p,1} \qquad \textit{for} \quad s > d. \tag{53}$$

Proof. Let $m_{-s}(x) = (1 + |x|^2)^{-s/2}$ and $g(x) = e^{-\pi x^2}$. Then for $f \in H^s_p$ we have:

$$\begin{aligned} V_g f(x,\omega) &= e^{-2\pi i x\omega}\, \mathcal{F}^{-1}(\hat{f} \cdot T_\omega \overline{\hat{g}})(x) \\ &= e^{-2\pi i x\omega}\, \mathcal{F}^{-1}(\hat{f} m_s \cdot m_{-s} T_\omega \overline{\hat{g}})(x) \\ &= e^{-2\pi i x\omega} \left(\mathcal{F}^{-1}(\hat{f} m_s) * \mathcal{F}^{-1}(m_{-s} \cdot T_\omega \overline{\hat{g}})(x)\right). \end{aligned}$$

Hence, by Young's convolution inequality,

$$\|V_g f(\cdot,\omega)\|_{L^p} \leq \|\mathcal{F}^{-1}(\hat{f} m_s)\|_{L^p}\, \|\mathcal{F}^{-1}(m_{-s} \cdot T_\omega \overline{\hat{g}})\|_{L^1},$$

and so

$$\begin{aligned} \|V_g f\|_{L^{p,1}} &\leq \|f\|_{H^s_p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}^{-1}(m_{-s} \cdot T_\omega \overline{\hat{g}})(x)|\, dx\, d\omega \\ &= \|f\|_{H^s_p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_{\hat{g}} m_{-s}(\omega, -x)|\, dx\, d\omega \\ &= \|f\|_{H^s_p}\, \|V_g \check{m}_{-s}\|_{L^1}. \end{aligned} \tag{54}$$

Using (39), we have that $\|V_g\check{m}_{-s}\|_{L^1} < \infty$ for $s > d$, which concludes the proof. $\square$

The next corollary holds only in dimension one, and gives a useful sufficient condition on a function to be in M^1. In particular, (55) below gives a new proof of a conjecture of Feichtinger, that $W^{2,1} \subset M^1$ when $d = 1$. We point out that another (unpublished) proof of this conjecture was obtained by Gröchenig. The corollary follows from the identification of the Bessel potential space H_p^2 with the Sobolev space $W^{2,p}$ obtained by imposing that f and its first two (distributional) derivatives belong to L^p ($p = 1$ or $p = \infty$).

Before proving this corollary, we present the proof of the identification of H_p^2 and $W^{2,p}$ when $p = 1$ or $p = 2$ and $d = 1$. We refer to [55, Sect. 6.6], and [9, Theorem 16] for more details on these identifications.

Lemma 4.2.6. *If $p = 1$ or $p = \infty$, let*

$$W^{2,p}(\mathbb{R}) = \{f \in L^p(\mathbb{R}) : f', f'' \in L^p(\mathbb{R}), \|f\|_{W^{2,p}} = \sum_{k=0}^{2} \|f^{(k)}\|_p < \infty\}.$$

Then

$$H_p^2 = W^{2,p}$$

Proof. Case I: Assume $p = 1$. Let $f \in W^{2,1}(\mathbb{R})$, then $f, f', f'' \in L^1(\mathbb{R})$. Moreover, $\hat{f}''(\omega) = -4\pi^2 \hat{f}(\omega)$. Thus,

$$\begin{aligned} \mathcal{F}^{-1}((1 + |\omega|^2)\hat{f}) &= \mathcal{F}^{-1}(\hat{f}) + \mathcal{F}^{-1}(\omega^2 \hat{f}) \\ &= f - \frac{1}{4\pi^2} f''. \end{aligned}$$

Consequently,

$$\mathcal{F}^{-1}((1 + |\omega|^2)\hat{f}) \in L^1(\mathbb{R}),$$

and moreover,

$$\begin{aligned}\|\mathcal{F}^{-1}((1+|\omega|^2)\hat{f})\|_{L^1} &\leq \|f\|_{L^1} + \frac{1}{4\pi^2}\,\|f''\|_{L^1}\\ &\leq C\sum_{k=0}^{2}\|f^{(k)}\|_{L^1}.\end{aligned}$$

Thus,

$$W^{2,1} \subset H_1^2.$$

For the converse, let h be the function defined on $\mathbb{R}$, such that

$$h(x) = \begin{cases} e^{-x} & : \quad x \geq 0, \\ 0 & : \quad x < 0. \end{cases}$$

We easily obtain that for $\omega \in \mathbb{R}$, $\hat{h}(\omega) = \frac{1}{1+2\pi i\omega}$.

Now let $f \in H_1^2(\mathbb{R})$, then $f \in L^1(\mathbb{R})$ and $\mathcal{F}^{-1}((1+\omega^2)\hat{f}) \in L^1$. Thus, $g = \mathcal{F}^{-1}(\omega^2\,\hat{f}) \in L^1$. Moreover, $f \in L^1(\mathbb{R})$ implies that $f \in \mathcal{S}'(\mathbb{R})$, and so f'' exists as an element of $\mathcal{S}'(\mathbb{R})$, hence $\hat{f}''(\omega) = -4\pi^2\,\omega^2\,\hat{f}$, where the equality holds in a distributional sense. Thus, $\hat{f}'' = \hat{g}$, and $g \in L^1(\mathbb{R})$, therefore the uniquness of the Fourier transform implies that $f'' = g \in L^1(\mathbb{R})$.

We now show that $f, f'' \in L^1(\mathbb{R})$ implies that f' which exists as a distribution is in $L^1(\mathbb{R})$. To that end, note that from the fact that $f, f'', h \in L^1(\mathbb{R})$, we obtain that $f * h$, and $f'' * h$ are both L^1 functions, and we have the following equalities:

$$\begin{aligned}\widehat{h * f''}(\omega) &= \frac{(-2\pi i\omega)^2}{1+2\pi i\omega}\,\hat{f}(\omega)\\ &= (-1 + 2\pi i\omega + \frac{1}{1+2\pi i\omega})\,hatf(\omega)\\ &= -\hat{f}(\omega) + 2\pi i\omega\,\hat{f}(\omega) + \frac{\hat{f}(\omega)}{1+2\pi i\omega}.\end{aligned}$$

Consequently,

$$h * f'' = -f + f * h + k,$$

where $k = \mathcal{F}^{-1}(2\pi i\omega, \hat{f})$. The last equation also implies that $k \in L^1(\mathbb{R})$, and moreover,

$$\hat{k}(\omega) = 2\pi i\omega \hat{f}(\omega) = \hat{f}'(\omega).$$

Using again the uniqueness of the Fourier transform, we conclude that $f' = k \in L^1(\mathbb{R})$, and moreover,

$$\|f'\|_{L^1} \leq 2\|f\|_{L^1} + \|f''\|_{L^1},$$

where we have used the fact that $\|h\|_{L^1} = 1$. Additionally, using the fact that $H_1^2 \subset L^1$, we conclude that

$$\sum_{k=0}^{2} \|f^{(k)}\|_{L^1} \leq C\,\|f\|_{H_1^2},$$

thus, $H_1^2 \subset W^{2,1}$, and the proof in this case is concluded.

The case $p = \infty$, is identical and so we omit it. □

Using the above lemma and Theorem 4.2.5 we have the following result in the case the dimension is $d = 1$.

Corollary 4.2.7. *If $d = 1$ and $p \in \{1, \infty\}$, then*

$$W^{2,p}(\mathbb{R}) \subset M^1(\mathbb{R}). \tag{55}$$

Proof. If $d = 1$ and $p \in \{1, \infty\}$, then from Lemma 4.2.6 we have that $H_p^2 = W^{2,p}$ and so the proof follows from Theorem 4.2.5. □

CHAPTER V

BILINEAR PSEUDODIFFERENTIAL OPERATORS ON MODULATION SPACES

In this chapter we present some applications of the modulation spaces. In particular, we study the boundedness of bilinear pseudodifferential operators on modulation spaces, as well as the boundedness of the linear Hilbert transform.

Modulation spaces have recently been used to formulate and prove boundedness results of linear pseudodifferential operators, which are formalisms that assign to a distribution a linear operator in such a way that properties of the distribution can be inferred from properties of the corresponding operator. The Weyl and the Kohn-Nirenberg correspondences are well-known examples of pseudodifferential operators, which can be expressed as a superposition of time-frequency shifts. In particular, if $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ the Weyl correspondence associate to it the operator $T_\sigma : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d)$, such that

$$T_\sigma f = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, u)\, e^{-\pi i \xi \cdot u}\, T_{-u} M_\xi \, du\, d\xi,$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. Thus, because the operator can be realized as superposition of time-frequency shifts, the modulation spaces appear to be natural spaces in which to formulate and prove boundedness results of such operators. We refer to [45, 42, 41], for more details on the recent developments of pseudodifferential operators in the realm of the modulation spaces.

In the first section of the present chapter, we deal with bilinear integral operators (defined by a non-smooth kernel) on modulation spaces. This class of operators is large enough to include the bilinear pseudodifferential operators with non-smooth symbols. In particular, we prove that symbols in the Feichtinger algebra give rise to bounded bilinear pseudodifferential operators. We refer to [11, 13, 12, 49] for background and more detail about these operators.

The second section is devoted to the boundedness of the linear Hilbert transform on the modulation spaces defined on the real line. We use a discrete approach to study the Hilbert transform, and rely on its L^2 theory to some extent.

5.1 Bilinear operators on modulation spaces

5.1.1 Definition and background

A bilinear pseudodifferential operator T_σ is á priori defined through its (distributional) symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{3d})$ as a mapping from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ by:

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(x,\xi,\eta)\,\hat{f}(\xi)\,\hat{g}(\eta)\,e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta, \tag{56}$$

for $f, g \in \mathcal{S}(\mathbb{R}^d)$. A natural problem then is to find sufficient (nontrivial) conditions on the symbol that ensure the boundedness of the operator on products of certain Banach spaces such as Lebesgue, Sobolev, or Besov spaces [11, 13, 12, 35, 36]. For instance, it is known that the condition

$$|\partial_x^\alpha\,\partial_\xi^\beta\,\partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha,\beta,\gamma}\,(1+|\xi|+|\eta|)^{-|\beta|-|\gamma|} \tag{57}$$

for $(x,\xi,\eta) \in \mathbb{R}^{3d}$ and all multi-indices α, β, γ is enough to prove the boundedness of the operator defined by (56) from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$ when $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $p, q > 1$. This result was first obtained by Coifman and Meyer [11], [13], [12], who noticed that, in general, if the symbol is smooth and has certain decay, then the boundedness of the corresponding operator can be studied through its decomposition into elementary operators via techniques related to Littlewood-Paley theory. Grafakos and Torres [35] used the wavelet expansions of the Triebel-Lizorkin spaces that were proved by Frazier and Jawerth [29, 30] to decompose instead the function on which the operator acts, and thereby converting the boundedness question into the boundedness of an infinite matrix. By imposing some convenient decay conditions on the entries of the corresponding matrix they obtain some boundedness results on the operator side on products Triebel-Lizorkin spaces. Here again, the symbols of the operators are assumed to be sufficiently smooth and to have decay at infinity.

In this section, we use Gabor expansions of tempered distributions in the modulation spaces to prove the boundedness of bilinear integral operators with non-smooth kernels, of which (56) will be shown to be a particular case.

Throughout this chapter, ω_s will denote the submultiplicative weight function defined on $\mathbb{R}^{2d}$ by $\omega_s(x,y) = (1+|x|^2+|y|^2)^{s/2}$. Moreover, we let Ω_s denote the extension of ω_s on $\mathbb{R}^{6d}$ given by

$$\Omega_s(X,Y) = (\omega_s \otimes \omega_s \otimes \omega_s)(X,Y) = \omega_s(x_1,x_2)\,\omega_s(x_3,y_1)\,\omega_s(y_2,y_3),$$

where $X=(x_1,x_2,x_3), Y=(y_1,y_2,y_3) \in \mathbb{R}^{3d}$. If A is an invertible operator on $\mathbb{R}^{6d}$ we denote $\tilde{\Omega}_s^{\,A}$ the weight function defined by

$$\tilde{\Omega}_s^A(X,Y) = \Omega_s(A(X,Y)),$$

where $X,Y \in \mathbb{R}^{3d}$. Additionally, we define $\tilde{\omega}_s$ on $\mathbb{Z}^{2d}$ by $\tilde{\omega}_s(l,k) = \omega_s(\alpha l, \beta k)$ for $\alpha, \beta > 0$, and $\widetilde{\Omega_s}$ is defined similarly.

Before considering general bilinear integral operators, we state a result which characterizes the modulation space $M^1_{\Omega_s}(\mathbb{R}^{3d})$ in terms of Gabor frames using standard tensor product arguments; see [41, p. 272] for further details.

Proposition 5.1.1. *Let $\phi \in M^1_{\omega_s}(\mathbb{R}^d)$ be such that $\{M_{\beta n}T_{\alpha k}\phi\}_{k,n\in\mathbb{Z}^d}$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with (canonical) dual $\gamma \in M^1_{\omega_s}(\mathbb{R}^d)$. Then $\mathcal{K} \in M^1_{\Omega_s}$ if and only if*

$$\mathcal{K} = \sum_{k,m,i,l,n,j\in\mathbb{Z}^d} \langle \mathcal{K}, M_{\beta n}T_{\alpha m}\gamma \otimes M_{\beta l}T_{\alpha k}\overline{\gamma} \otimes M_{\beta j}T_{\alpha i}\overline{\gamma}\rangle\, M_{\beta n}T_{\alpha m}\phi \otimes M_{\beta l}T_{\alpha k}\overline{\phi} \otimes M_{\beta j}T_{\alpha i}\overline{\phi}$$

with unconditional convergence of the series in $M^1_{\Omega_s}(\mathbb{R}^{3d})$. Moreover, the norm of $\mathcal{K}$ in $M^1_{\Omega_s}$ is equivalent to the norm of its sequence of Gabor coefficients $\big(\langle \mathcal{K}, M_{\beta n}T_{\alpha m}\gamma \otimes M_{\beta l}T_{\alpha k}\overline{\gamma} \otimes M_{\beta j}T_{\alpha i}\overline{\gamma}\rangle\big)_{k,m,i,l,n,j\in\mathbb{Z}^d}$ in $\ell^1_{\tilde{\Omega}_s}(\mathbb{Z}^{6d})$.

5.1.2 Bilinear operators

Definition 5.1.2. A bilinear operator associated with a kernel $K \in \mathcal{S}'(\mathbb{R}^{3d})$, is a mapping B_K defined á priori from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ by

$$B_K(f,g)(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y,z)\, f(y)\, g(z)\, dy\, dz, \tag{58}$$

for $f, g \in \mathcal{S}(\mathbb{R}^d)$.

One of our objectives in this section is to study the boundedness of (58) on products of modulation spaces, and to derive from such results the boundedness of (56). The next proposition establishes the relationship between a bilinear integral operator and a bilinear pseudodifferential operator defined by (56). We define an operator U acting on functions defined on $\mathbb{R}^{3d}$ by

$$Uf(x,y,z) = f(x, y-x, z-x).$$

It easy to check that U is a unitary operator on L^2, is an isomorphism on $\mathcal{S}$, and extends to an isomorphism on $\mathcal{S}'$. Moreover,

$$U^* f(x,y,z) = U^{-1} f(x,y,z) = f(x, y+x, z+x).$$

Proposition 5.1.3. *Let T_σ be a bilinear pseudodifferential operator associated to a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{3d})$ defined by* (56)*. Then T_σ is a bilinear integral operator B_K with kernel $K(x,y,z) = U\mathcal{F}_1^{-1}\hat{\sigma}(x,y,z)$, where $\mathcal{F}_1^{-1}$ denotes the inverse Fourier transform in the first variable, and U is the operator defined above.*

Proof. For $f, g \in \mathcal{S}$ we have:

$$\begin{aligned} T_\sigma(f,g)(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(x,\xi,\eta)\, \hat{f}(\xi)\, \hat{g}(\eta)\, e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta \\ &= \iiiint \sigma(x,\xi,\eta)\, f(y)\, g(z)\, e^{-2\pi i \xi\cdot y}\, e^{-2\pi i \eta\cdot z}\, e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta\, dy\, dz \\ &= \iint K(x,y,z)\, f(y)\, g(z)\, dy\, dz = B_K(f,g)(x), \end{aligned}$$

where

$$K(x,y,z) = \int \int \sigma(x,\xi,\eta) e^{-2\pi i \xi\cdot(y-x)}\, e^{-2\pi i \eta\cdot(z-x)}\, d\xi\, d\eta$$

$$= \mathcal{F}_2\mathcal{F}_3\sigma(x, y-x, z-x)$$

$$= U\mathcal{F}_1^{-1}\hat{\sigma}(x,y,z).$$

Here, $\mathcal{F}_j$ denotes the Fourier transform in the j^{th} variable. □

We show in the next proposition that the symbol of the bilinear pseudodifferential operator is in $M^1_{\Omega_s^B}$ if and only if the corresponding integral kernel as defined in Proposition 5.1.3 is in $M^1_{\Omega_s}$, where B is the invertible transformation defined on $\mathbb{R}^{6d}$ defined by

$$B(X,Y) = (x_1, x_1+y_2, x_1+y_3, x_2+x_3+y_1+y_3, -x_2, -x_3),$$

for $X = (x_1,x_2,x_3), Y = (y_1,y_2,y_3) \in \mathbb{R}^{3d}$.

Proposition 5.1.4. *$\sigma \in M^1_{\Omega_s^B}(\mathbb{R}^{3d})$ if and only if $K = U\mathcal{F}_1^{-1}\hat{\sigma} \in M^1_{\Omega_s}(\mathbb{R}^{3d})$.*

Proof. Let $G \in \mathcal{S}(\mathbb{R}^{3d})$. For $u = (u_1,u_2,u_3)$, and $v = (v_1,v_2,v_3) \in \mathbb{R}^{3d}$ we have $V_G\sigma(u,v) = \langle \sigma, M_vT_uG\rangle = \langle \mathcal{F}^{-1}\mathcal{F}_1^{-1}U^*K, M_vT_uG\rangle = \langle K, U\mathcal{F}_1^{-1}\mathcal{F}M_vT_uG\rangle$. Hence,

$$\begin{aligned} V_G\sigma(u,v) &= e^{-2\pi i(u_2\cdot v_2+u_3\cdot v_3)}\,\langle K, M_{(v_1+u_2+u_3,-u_2,-u_3)}T_{(u_1,v_2+u_1,v_3+u_1)}H\rangle \\ &= e^{-2\pi i(u_2\cdot v_2+u_3\cdot v_3)}\,V_HK(u_1, v_2+u_1, v_3+u_1, v_1+u_2+u_3, -u_2, -u_3) \\ &= e^{-2\pi i(u_2\cdot v_2+u_3\cdot v_3)}\,V_HK(B(u,v)), \end{aligned} \tag{59}$$

where $H = U\mathcal{F}_1^{-1}\hat{G}$. Consequently, we have

$$|V_G\sigma(u,v)| = |V_HK(B(u,v))|.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^{3d}}\int_{\mathbb{R}^{3d}} |V_HK(u,v)|\Omega_s(u,v)\,du\,dv &= \int_{\mathbb{R}^{3d}}\int_{\mathbb{R}^{3d}} |V_G\sigma(B^{-1}(u,v))|\Omega_s(u,v)\,du\,dv \\ &= \int_{\mathbb{R}^{3d}}\int_{\mathbb{R}^{3d}} |V_G\sigma(u,v))|\Omega_s^B(u,v)\,du\,dv. \end{aligned}$$

Thus

$$\|K\|_{M^1_{\Omega_s}} = \|\sigma\|_{M^1_{\Omega_s^B}},$$

and the proof is complete. □

5.1.3 A discrete model

Consider $\phi \in \mathcal{S}(\mathbb{R}^d)$ that generates a Gabor frame for L^2 with (canonical) dual $\gamma \in \mathcal{S}(\mathbb{R}^d)$. We can then expand f, g and h in $\mathcal{S}(\mathbb{R}^d)$ as in Theorem 3.8, where the series converge unconditionally in every modulation space norm as long as $p, q \neq \infty$. Then using (58), we obtain:

$$\langle B_K(f,g),h\rangle = \iiint_{\mathbb{R}^{3d}} K(x,y,z) \sum_{k,l\in\mathbb{Z}^d} \langle f, M_{\beta l}T_{\alpha k}\gamma\rangle\, M_{\beta l}T_{\alpha k}\phi(y) \times$$

$$\sum_{m,n\in\mathbb{Z}^d} \langle g, M_{\beta n}T_{\alpha m}\gamma\rangle\, M_{\beta n}T_{\alpha m}\phi(z)\, \overline{\sum_{i,j\in\mathbb{Z}^d} \langle h, M_{\beta j}T_{\alpha i}\gamma\rangle\, M_{\beta j}T_{\alpha i}\phi(x)}\, dx\, dy\, dz$$

$$= \sum_{i,j}\sum_{k,l}\sum_{m,n} \langle f, M_{\beta l}T_{\alpha k}\gamma\rangle\, \langle g, M_{\beta n}T_{\alpha m}\gamma\rangle\, \overline{\langle h, M_{\beta j}T_{\alpha i}\gamma\rangle} \times$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y,z)\, \overline{M_{\beta j}T_{\alpha i}\phi(x)} M_{\beta l}T_{\alpha k}\phi(y)\, M_{\beta n}T_{\alpha m}\phi(z)\, dx\, dy\, dz$$

$$= \sum_{i,j}\sum_{k,n}\sum_{l,m} \langle f, M_{\beta l}T_{\alpha k}\gamma\rangle\, \langle g, M_{\beta n}T_{\alpha m}\gamma\rangle\, \overline{\langle h, M_{\beta j}T_{\alpha i}\gamma\rangle} \times$$

$$\langle B_K(M_{\beta l}T_{\alpha k}\phi, M_{\beta n}T_{\alpha m}\phi), M_{\beta j}T_{\alpha i}\phi\rangle. \tag{60}$$

The exchange of the integrals and summations above is justified since $f, g, h \in \mathcal{S}$ have absolutely summable Gabor coefficients. Moreover, $K \in \mathcal{S}'(\mathbb{R}^{3d}) = \bigcup_{s\geq 0} M^\infty_{1/\Omega_s}$ (see Proposition 2.3.9), and $\phi \in \mathcal{S}$ implies that the triple integral in the second equality is uniformly bounded with respect to $i, j, k, l, m, n \in \mathbb{Z}^d$. More precisely, define

$$M_{\beta(j,l,n)}T_{\alpha(i,k,m)}\Phi(x,y,z) = M_{\beta j}T_{\alpha i}\phi(x)\, \overline{M_{\beta l}T_{\alpha k}\phi(y)\, M_{\beta n}T_{\alpha m}\phi(z)}.$$

Clearly, $M_{\beta(j,l,n)}T_{\alpha(i,k,m)}\Phi \in \mathcal{S}(\mathbb{R}^{3d}) \subset M^1_{\Omega_s}$ for all $s \geq 0$. Moreover, $K \in M^\infty_{1/\Omega_s}$ where $s > 0$. Thus, using the fact that the time-frequency shift operator acts isometrically on $M^1_{\omega_s}$, we have

$$\begin{aligned}
&\Big|\iiint_{\mathbb{R}^{3d}} K(x,y,z)\,\overline{M_{\beta j}T_{\alpha i}\phi(x)}M_{\beta l}T_{\alpha k}\phi(y)\,M_{\beta n}T_{\alpha m}\phi(z)\,dx\,dy\,dz\Big| \\
\leq\ & \iiint_{\mathbb{R}^{3d}} |K(x,y,z)|\,|M_{\beta(j,l,n)}T_{\alpha(i,k,m)}\Phi(x,y,z)|\,dx\,dy\,dz \\
\leq\ & \|K\|_{M^\infty_{1/\Omega_s}}\,\|M_{\beta(j,l,n)}T_{\alpha(i,k,m)}\Phi\|_{M^1_{\Omega_s}} \\
=\ & \|K\|_{M^\infty_{1/\Omega_s}}\,\|\phi\|^3_{M^1_{\omega_s}}.
\end{aligned}$$

Therefore, to study the boundedness of B_K on products of modulation spaces, it suffices to analyze the boundedness of the matrix $B = (b_{ij,kl,mn})$ defined by

$$b_{ij,kl,mn} = \langle B_K(M_{\beta l}T_{\alpha k}\phi, M_{\beta n}T_{\alpha m}\phi), M_{\beta j}T_{\alpha i}\phi\rangle \tag{61}$$

on products of appropriate sequence spaces.

The next theorem will be of special importance in proving our main results. In particular, it shows that, under some mild condition on its entries, an infinite matrix yields a bounded operator on products of sequence spaces associated with the modulation spaces. For an infinite matrix $(a_{mn,ij,kl})$, let $\mathcal{O}$ denote the bilinear operator associated to it, i.e.,

$$(\mathcal{O}(f_{ij}),(g_{kl}))_{mn} = \sum_{ij,kl} a_{mn,ij,kl}\, f_{ij}\, g_{kl},$$

where (f_{ij}) and (g_{kl}) are sequences defined on $\mathbb{Z}^{2d}$.

Theorem 5.1.5. *Let ν be an s-moderate weight, and let $1 \leq p_i, q_i, r_i < \infty$ for $i = 1, 2$ be such that $\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}$. If $(a_{mn,ij,kl}) \in \ell^1_{\tilde{\Omega}_s}(\mathbb{Z}^{6d})$, then $\mathcal{O}$ is a bounded operator from $\ell^{p_1,p_2}_{\tilde{\nu}}(\mathbb{Z}^{2d}) \times \ell^{q_1,q_2}_{\tilde{\nu}}(\mathbb{Z}^{2d})$ into $\ell^{r_1,r_2}_{\tilde{\nu}}(\mathbb{Z}^{2d})$. In particular, if $(a_{mn,ij,kl}) \in \ell^1(\mathbb{Z}^{6d})$ then $\mathcal{O}$ is a bounded operator from $\ell^{p_1,p_2}(\mathbb{Z}^{2d}) \times \ell^{q_1,q_2}(\mathbb{Z}^{2d})$ into $\ell^{r_1,r_2}(\mathbb{Z}^{2d})$.*

Proof. Let $(f_{ij}) \in \ell^{p_1,p_2}_{\tilde{\nu}}(\mathbb{Z}^{2d})$, $(g_{kl}) \in \ell^{q_1,q_2}_{\tilde{\nu}}(\mathbb{Z}^{2d})$ and $(h_{mn}) \in \ell^{r_1',r_2'}_{1/\tilde{\nu}}(\mathbb{Z}^{2d})$ where r_1', r_2'

are the dual indices of r_1, respectively r_2. We have

$$\begin{aligned}
|\langle \mathcal{O}((f_{ij}),(g_{kl})),(h_{mn})\rangle| &\leq \sum_{m,n,i,j,k,l} |a_{mn,kl,ij}|\,|f_{ij}|\,|g_{kl}|\,|h_{mn}| \\
&= \sum_{m,n,i,j,k,l} |a_{mn,kl,ij}|^{\frac{1}{p_1}}\,|f_{ij}|\,\frac{\tilde{\nu}(i,j)}{\tilde{\nu}(i,j)}\,|a_{mn,kl,ij}|^{\frac{1}{q_1}}\,|g_{kl}|\,\frac{\tilde{\nu}(k,l)}{\tilde{\nu}(k,l)}\times \\
&\qquad |a_{mn,kl,ij}|^{\frac{1}{r_1'}}\,|h_{mn}|\,\frac{\tilde{\nu}(m,n)}{\tilde{\nu}(m,n)} \\
&\leq C^3 \sum_{m,n,i,j,k,l} |a_{mn,kl,ij}|^{\frac{1}{p_1}}\,\tilde{\nu}(i,j)\,|f_{ij}|\,\tilde{\omega_s}(i,j)\times \\
&\qquad |a_{mn,ij,kl}|^{\frac{1}{q_1}}\,\tilde{\nu}(k,l)\,|g_{kl}|\,\tilde{\omega_s}(k,l)\times \\
&\qquad |a_{mn,ij,kl}|^{\frac{1}{r_1'}}\,\frac{1}{\tilde{\nu}(m,n)}\,|h_{mn}|\,\tilde{\omega_s}(m,n) \\
&= C^3 \sum_{m,n,i,j,k,l} \big(|\tilde{a}_{mn,kl,ij}|\big)^{1/p_1}\,|f_{ij}|\,\tilde{\nu}(i,j)\times \\
&\qquad \big(|\tilde{a}_{mn,kl,ij}|\big)^{1/q_1}\,|g_{kl}|\,\tilde{\nu}(k,l)\times \\
&\qquad \big(|\tilde{a}_{mn,kl,ij}|\big)^{1/r_1'}\,|h_{mn}|\,\frac{1}{\tilde{\nu}(m,n)},
\end{aligned}$$

where

$$\tilde{a}_{mn,kl,ij} = a_{mn,kl,ij}\,\tilde{\Omega}_s(m,n,k,l,i,j) = a_{mn,kl,ij}\,\tilde{\omega}_s(m,n)\,\tilde{\omega}_s(k,l)\,\tilde{\omega}_s(i,j).$$

We have used the fact that ν, and $\frac{1}{\nu}$ are s-moderate with the same constant C. Since $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{r_1}$, or equivalently $\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1'}$, we can apply Hölder's inequality to obtain the following:

$$\begin{aligned}
|\langle \mathcal{O}((f_{ij}),(g_{kl})),(h_{mn})\rangle| &\leq C^3 \Big(\sum_{m,n,i,j,k,l} |\tilde{a}_{mn,ij,kl}|\,|f_{ij}|^{p_1}\,\tilde{\nu}(i,j)^{p_1}\Big)^{\frac{1}{p_1}}\times \\
&\qquad \Big(\sum_{m,n,i,j,k,l} |\tilde{a}_{mn,ij,kl}|\,|g_{kl}|^{q_1}\,\tilde{\nu}(k,l)^{q_1}\Big)^{\frac{1}{q_1}}\times
\end{aligned}$$

$$\Big(\sum_{m,n,i,j,k,l} |\tilde{a}_{mn,ij,kl}|\,|h_{mn}|^{r_1'}\,\frac{1}{\tilde{\nu}(m,n)^{r_1'}}\Big)^{\frac{1}{r_1'}}$$

$$\le C^3 \Big(\sup_{i,j}|f_{ij}|\,\tilde{\nu}(i,j)\Big)\Big(\sup_{k,l}|g_{kl}|\,\tilde{\nu}(k,l)\Big)\times$$

$$\Big(\sup_{m,n}|h_{mn}|\,\frac{1}{\tilde{\nu}(m,n)}\Big)\Big(\sum_{m,i,k}\sum_{n,j,l}|\tilde{a}_{mn,ij,kl}|\Big)$$

$$\le C^3\,\|a_{mn,ij,kl}\|_{\ell_{\tilde{\Omega}_s^1}}\Big(\sum_i\Big(\sum_j|f_{ij}|^{p_1}\,\tilde{\nu}(i,j)^{p_1}\Big)^{\frac{p_2}{p_1}}\Big)^{\frac{1}{p_2}}\times$$

$$\Big(\sum_k\Big(\sum_l|g_{kl}|^{q_1}\,\tilde{\nu}(k,l)^{q_1}\Big)^{\frac{q_2}{q_1}}\Big)^{\frac{1}{q_2}}\times$$

$$\Big(\sum_m\Big(\sum_n|h_{mn}|^{r_1'}\,\frac{1}{\tilde{\nu}(m,n)^{r_1'}}\Big)^{\frac{r_2'}{r_1'}}\Big)^{\frac{1}{r_2'}}$$

$$\le C^3\,\|a_{mn,ij,kl}\|_{\ell^1_{\tilde{\Omega}_s}}\,\|f_{i,j}\|_{\ell^{p_1,p_2}_{\tilde{\nu}}}\,\|g_{k,l}\|_{\ell^{q_1,q_2}_{\tilde{\nu}}}\,\|h_{m,n}\|_{\ell^{r_1',r_2'}_{1/\tilde{\nu}}},$$

where we have used the fact that $\ell^{p,q}(\mathbb{Z}^{2d})\subset\ell^\infty(\mathbb{Z}^{2d})$, i.e.,

$$\sup_{m,n}|x_{m,n}|\le\Big(\sum_{n\in\mathbb{Z}^d}\Big(\sum_{m\in\mathbb{Z}^d}|x_{m,n}|^p\Big)^{q/p}\Big)^{1/q}.$$

Moreover, using the duality of the $\ell^{p,q}_{\tilde{\nu}}$-spaces i.e.,

$$\|a\|_{\ell^{r_1,r_2}_{\tilde{\nu}}}=\sup_{\|b\|_{\ell^{r_1',r_2'}_{1/\tilde{\nu}}}=1}\sum_{m,n\in\mathbb{Z}^d}|a_{m,n}|\,|b_{m,n}|,$$

we get that

$$\|\mathcal{O}((f_{ij}),(g_{kl}))\|_{\ell^{r_1,r_2}_{\tilde{\nu}}}\le C^3\,\|a_{mn,ij,kl}\|_{\ell^1_{\tilde{\Omega}_s}}\,\|(f_{ij})\|_{\ell^{p_1,p_2}_{\tilde{\nu}}}\,\|(g_{kl})\|_{\ell^{q_1,q_2}_{\tilde{\nu}}}.$$

The second part of the theorem follows by choosing $\nu=\omega_0\equiv 1$. □

5.1.4 Boundedness of bilinear pseudodifferential operators

Our first main result of this section shows that a bilinear integral operator with kernel in the modulation space $M^1_{\Omega_s}$ — in particular, in the Feichtinger algebra — gives rise to a bounded operator.

Theorem 5.1.6. *Let ν be an s-moderate weight, and let $1 \leq p_i, q_i, r_i < \infty$ for $i = 1, 2$ be such that $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{r_1}$. If $K \in M^1_{\Omega_s}(\mathbb{R}^{3d})$, then the bilinear integral operator B_K defined by (58) can be extended as a bounded operator from $M^{p_1,p_2}_\nu(\mathbb{R}^d) \times M^{q_1,q_2}_\nu(\mathbb{R}^d)$ into $M^{r_1,r_2}_\nu(\mathbb{R}^d)$.*

Proof. Let $f, g, h \in \mathcal{S}(\mathbb{R}^d)$ and expand each of these functions into their Gabor series, i.e., $f = \sum_{i,j} \langle f, M_{\beta j} T_{\alpha i} \phi \rangle \, M_{\beta j} T_{\alpha i} \gamma$, $g = \sum_{k,l} \langle g, M_{\beta l} T_{\alpha k} \phi \rangle \, M_{\beta l} T_{\alpha k} \gamma$, and $h = \sum_{m,n} \langle h, M_{\beta n} T_{\alpha m} \phi \rangle \, M_{\beta n} T_{\alpha m} \gamma$, where ϕ and γ are dual Gabor frames as in Theorem 2.3.11. By Proposition 5.1.1, the matrix defined by (61) belongs to $\ell^1_{\tilde{\Omega}_s}$ since $K \in M^1_{\Omega_s}$. Therefore, using Theorem 5.1.5 we have the following estimates:

$$
\begin{aligned}
|\langle B_K(f,g), h \rangle| &= |\sum_{mn} \sum_{ij} \sum_{kl} a_{mn,ij,kl} \, \langle f, M_{\beta j} T_{\alpha i} \phi \rangle \, \langle g, M_{\beta l} T_{\alpha k} \phi \rangle \, \overline{\langle h, M_{\beta n} T_{\alpha m} \phi \rangle}| \\
&\leq C \, \|a_{mn,ij,kl}\|_{\ell^1_{\tilde{\Omega}_s}} \, \|\langle f, M_{\beta j} T_{\alpha i} \phi \rangle\|_{\ell^{p_1,p_2}_{\tilde{\nu}}} \times \\
&\qquad \|\langle g, M_{\beta l} T_{\alpha k} \phi \rangle\|_{\ell^{q_1,q_2}_{\tilde{\nu}}} \, \|\langle h, M_{\beta n} T_{\alpha m} \phi \rangle\|_{\ell^{r_1',r_2'}_{1/\tilde{\nu}}} \\
&\leq C \, \|K\|_{M^1_{\Omega_s}} \, \|f\|_{M^{p_1,p_2}_\nu} \, \|g\|_{M^{q_1,q_2}_\nu} \, \|h\|_{M^{r_1',r_2'}_{1/\nu}},
\end{aligned}
$$

by duality we obtain

$$
\|B_K(f,g)\|_{M^{r_1,r_2}_\nu} \leq C \, \|K\|_{M^1_{\Omega_s}} \, \|f\|_{M^{p_1,p_2}_\nu} \, \|g\|_{M^{q_1,q_2}_\nu}.
$$

The result then follows by standard density arguments, using the fact that $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}_\nu$ for $1 \leq p, q < \infty$. □

The previous result together with Propositions 5.1.3 and 5.1.4 yields our second main result of this chapter, which provides a sufficient condition on the symbol so that the operator (56) is bounded on products of modulation spaces. Recall that the invertible transformation B was defined on $\mathbb{R}^{6d}$ by

$$
B(X, Y) = (x_1, x_1 + y_2, x_1 + y_3, x_2 + x_3 + y_1 + y_3, -x_2, -x_3).
$$

Theorem 5.1.7. *Let ν be an s-moderate weight, and let $1 \leq p_i, q_i, r_i < \infty$ for $i = 1, 2$ be such that $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{r_1}$. If $\sigma \in M^1_{\Omega_s^B}(\mathbb{R}^{3d})$, then the bilinear pseudodifferential operator T_σ defined by (56) can be extended to a bounded operator from $M_\nu^{p_1,p_2}(\mathbb{R}^d) \times M_\nu^{q_1,q_2}(\mathbb{R}^d)$ into $M_\nu^{r_1,r_2}(\mathbb{R}^d)$.*

Proof. By Proposition 5.1.4, $\sigma \in M^1_{\Omega_s^B}$ if and only if $K \in M^1_{\Omega_s}$, where K is the kernel of the corresponding integral operator, and the result follows from Theorem 5.1.6. □

If we assume that $\nu = \omega_0 \equiv 1$, and that $p_1 = p_2 = p$ and $q_1 = q_2 = q$ (hence $r_1 = r_2 = r$), we obtain the following.

Corollary 5.1.8. *Let $2 \leq p, q < \infty$ and $1 \leq r \leq 2$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $\sigma \in M^1(\mathbb{R}^{3d})$, then T_σ can be extended to a bounded operator from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$. In particular, if $\sigma \in M^1(\mathbb{R}^{3d})$, then T_σ has a bounded extension from $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Proof. If of $\leq p, q < \infty$ by Proposition 4.2.1 we have the following embeddings:

$$L^p \subset M^p, \qquad \text{and} \qquad L^q \subset M^q.$$

Thus, $L^p \times L^q \subset M^p \times M^q$. Moreover, since $1 \leq r \leq 2$, we have by the same proposition that $M^r \subset L^r$. These continuous embeddings combined with Theorem 5.1.7 imply then the result. □

Remark 5.1.9. It is remarkable that the condition $\sigma \in M^1(\mathbb{R}^{3d})$ does not necessarily imply any smoothness nor decay on the symbol. In particular, Coifman-Meyer-type conditions (57) are not necessarily satisfied by the symbols we consider.

Assume that $\nu(x, y) = \omega_s(x, y) = (1 + |x|^2 + |y|^2)^{s/2}$ for some $s > 0$, and that $p_i = q_i = 2$. Let ω_s^1 be the restriction of ω_s to $\mathbb{R}^d \times \{0\}$. Then the following holds.

Corollary 5.1.10. *If $\sigma \in M^1_{\Omega_s^B}$ then T_σ can be extended as a bounded bilinear pseudodifferential operator from $M^2_{\omega_s} \times M^2_{\omega_s}$ into $L^1_{\omega_s^1}$.*

Proof. Notice that $M^1_{\omega_s}$ is continuously embedded in $L^1_{\omega^1_s}$, cf. [41, Prop. 12.1.4]. So, we only need to prove that under the hypotheses of the corollary, the bilinear pseudodifferential operator can be extended to a bounded operator from $M^2_{\omega_s} \times M^2_{\omega_s}$ into $M^1_{\omega_s}$. This follows from Theorem 5.1.7 by taking $\nu = \omega_s$. □

Remark 5.1.11. a. If the symbol σ satisfies the estimates

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq C_{\beta,\gamma} (1 + |\xi| + |\eta|)^{-d-\epsilon} \tag{62}$$

for all $(x, \xi, \eta) \in \mathbb{R}^{3d}$, all multi-indices β and γ, and some $\epsilon > 0$, then it follows from [7, Theorem 1] that the corresponding bilinear pseudodifferential operator is bounded from $L^2 \times L^2$ into L^1. We wish to point out that, in general, neither that result nor Corollary 5.1.8 in this section imply each other. On one hand, if $g \in \mathcal{S}(\mathbb{R}^{2d})$ then $\sigma_1(x, \xi, \eta) = \chi_{[0,1[^d}(x) g(\xi, \eta)$, where $\chi_{[0,1[^d}$ is the characteristic function of the unit cube in $\mathbb{R}^d$, satisfies (62) and hence it yields a bounded operator from $L^2 \times L^2$ into L^1. However, because σ_1 is not a continuous function, it is not in $M^1(\mathbb{R}^{3d})$. Therefore, our corollary does not apply. On the other hand, functions in M^1 must be continuous, but there are non-differentiable functions in M^1, hence they do not satisfy (62), thus [7, Theorem 1] does not apply here.

b. Notice that (62) requires smoothness of the symbols only in the ξ and η variables whereas (57) imposes smoothness on all the variables x, ξ and η. Thus the two conditions are different.

5.2 Linear Hilbert transform on the modulation spaces

In this section, we consider the boundedness of the one-dimensional (linear) Hilbert transform — which is a prototypical example of a singular integral operator — on the modulation spaces. The Hilbert transform of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$Hf(x) = \tfrac{1}{\pi} \lim_{\epsilon \to 0} \int_{|t|>\epsilon} \tfrac{f(x-t)}{t}\, dt = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|x-t|>\epsilon} \tfrac{f(t)}{x-t}\, dt. \tag{63}$$

The boundedness of the Hilbert transform on the L^p spaces ($1 < p < \infty$) was established by M. Riesz using complex variable methods. The real variable method, initiated by Besicovitch and Titchmarsh, and further developed by Calderón and Zygmund, establishes that the Hilbert transform is of weak-type $(1, 1)$ (in fact, their theory applies to more general operators). More precisely, the following estimate holds:

$$|\{x \in \mathbb{R} : (Hf(x)) > \alpha\}| \leq \frac{C}{\alpha} \int_{\mathbb{R}} |f(x)|\, dx,$$

where C is a constant independent of f and $\alpha > 0$. Moreover, using a Fourier approach, it is easy to prove the boundedness of the Hilbert transform on L^2. Indeed, it is known that $\widehat{Hf}(\omega) = -i\, sign(\omega)\, \hat{f}(\omega)$, where *sign* denotes the sign function. Thus, using Plancherel's theorem we obtain

$$\begin{aligned} \|Hf\|_{L^2} &= \|\widehat{Hf}\|_{L^2} \\ &= \|\hat{f}\|_{L^2} \\ &= \|f\|_{L^2}, \end{aligned}$$

and this last equality proves the boundedness of H on L^2. The weak-type $(1, 1)$ result, and the L^2 boundedness together with duality and interpolation methods can be used to prove that H is bounded on L^p for $1 < p < \infty$. We refer to [55, Sect. 2] for more background on the Hilbert transform, and more generally, on the theory of singular integrals.

We prove the boundedness of H on the modulation spaces $M^{p,q}$, for $1 < p, q < \infty$, by converting the boundedness question into the boundedness of an infinite matrix acting on appropriate sequence spaces. We achieve this goal by expanding the function on which H acts into their Gabor expansions, and thus we are reduced to studying the boundedness of an associated infinite matrix on $\ell^{p,q}(\mathbb{Z}\times\mathbb{Z})$.

Let $\alpha,\beta>0$ be given, and let $\phi,\gamma\in\mathcal{S}(\mathbb{R})$ be such that $supp(\hat{\gamma})\subset(0,\beta)$ and $\hat{\gamma}\geq 0$. Assume, moreover, that $\{M_{\beta n}T_{\alpha k}\phi\}_{k,n}$, and $\{M_{\beta n}T_{\alpha k}\gamma\}_{k,n}$ are dual Gabor frames for $L^2(\mathbb{R})$. Then $\{M_{\beta n}T_{\alpha k}\phi\}_{k,n}$ and $\{M_{\beta n}T_{\alpha k}\gamma\}_{k,n}$ are also dual Gabor frames for all the modulation spaces $M^{p,q}(\mathbb{R})$ (see Theorem 2.3.11).

Proposition 5.2.1. *Let $f,g\in\mathcal{S}(\mathbb{R})$, then*

$$\langle Hf,g\rangle=\sum_{m,n\in\mathbb{Z}}\sum_{k,l\in\mathbb{Z}}\langle f,M_{\beta n}T_{\alpha m}\phi\rangle\,\overline{\langle g,M_{\beta l}T_{\alpha k}\phi\rangle}\,\langle HM_{\beta n}T_{\alpha m}\gamma,M_{\beta l}T_{\alpha k}\gamma\rangle. \tag{64}$$

Proof. If $f,g\in\mathcal{S}(\mathbb{R})$, then we can expand them into their Gabor expansions, i.e.,

$$f=\sum_{m,n\in\mathbb{Z}}\langle f,M_{\beta n}T_{\alpha m}\phi\rangle\,M_{\beta n}T_{\alpha m}\gamma,\quad\text{and}\quad g=\sum_{k,l\in\mathbb{Z}}\langle g,M_{\beta l}T_{\alpha k}\phi\rangle\,M_{\beta l}T_{\alpha k}\gamma,$$

with unconditional convergence in any modulation space. In particular, the assumptions on f,g,ϕ, and γ imply that the Gabor coefficients of f and g are absolutely summable. Then we have:

$$\begin{aligned}\langle Hf,g\rangle&=\int_{\mathbb{R}}Hf(x)\cdot\overline{g}(x)\,dx\\&=\int_{\mathbb{R}}\sum_{k,l\in\mathbb{Z}}\overline{\langle g,M_{\beta l}T_{\alpha k}\phi\rangle}\,Hf(x)\cdot\overline{M_{\beta l}T_{\alpha k}\gamma}(x)\,dx.\end{aligned}$$

Because H is bounded on L^2, we have that for each $k,n\in\mathbb{Z}$, $Hf\cdot\overline{M_{\beta l}T_{\alpha k}\gamma}\in L^1(\mathbb{R})$. Moreover, since $\langle g,M_{\beta l}T_{\alpha k}\phi\rangle\in\ell^1(\mathbb{Z}\times\mathbb{Z})$, we can apply Fubini's Theorem to obtain:

$$\langle Hf,g\rangle=\sum_{k,l\in\mathbb{Z}}\overline{\langle g,M_{\beta l}T_{\alpha k}\phi\rangle}\int_{\mathbb{R}}Hf(x)\cdot\overline{M_{\beta l}T_{\alpha k}\gamma}(x)\,dx. \tag{65}$$

Further, the adjoint H^* of H is bounded on L^2 (in fact, one can show that $H^*=$

$-H$), so we have the following:

$$\begin{aligned}\int_{\mathbb{R}} Hf(x)\,\overline{M_{\beta l}T_{\alpha k}\gamma}(x)\,dx &= \langle Hf, M_{\beta l}T_{\alpha k}\gamma\rangle \\ &= \langle f, H^{*}M_{\beta l}T_{\alpha k}\gamma\rangle \\ &= -\langle f, HM_{\beta l}T_{\alpha k}\gamma\rangle \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, M_{\beta n}T_{\alpha m}\phi\rangle\,\langle HM_{\beta n}T_{\alpha m}\gamma, M_{\beta l}T_{\alpha k}\gamma\rangle. \end{aligned} \tag{66}$$

We can now use (65) and (66) to obtain;

$$\langle Hf, g\rangle = \sum_{k,l}\sum_{m,n}\langle f, M_{\beta n}T_{\alpha m}\phi\rangle\,\overline{\langle g, M_{\beta l}T_{\alpha k}\phi\rangle}\,\langle HM_{\beta n}T_{\alpha m}\gamma, M_{\beta l}T_{\alpha k}\gamma\rangle. \tag{67}$$

□

We denote $A_{\alpha,\beta,\gamma}$ the sequence defined for $(k,l)\in\mathbb{Z}^2$ by

$$A_{\alpha,\beta,\gamma}(k,l) = |V_{\hat{\gamma}}\hat{\gamma}(\beta l, \alpha k)|.$$

The choice of the window γ implies in particular that $\gamma\in M^1$, hence

$$\sum_{l,k\in\mathbb{Z}} A_{\alpha,\beta,\gamma}(k,l) < \infty.$$

We can use the above proposition to prove the following.

Proposition 5.2.2. *If $f, g\in\mathcal{S}(\mathbb{R})$, then*

$$|\langle Hf, g\rangle| \leq \sum_{m,n\in\mathbb{Z}}\sum_{k,l\in\mathbb{Z}} A_{\alpha,\beta,\gamma}(m-k, l-n)\,|\langle f, M_{\beta n}T_{\alpha m}\phi\rangle|\,|\langle g, M_{\beta l}T_{\alpha k}\phi\rangle|. \tag{68}$$

Proof. Let S be the sign function defined by

$$S(x) = sign(x) = \begin{cases} -1 & : \quad x<0, \\ 0 & : \quad x=0, \\ +1 & : \quad x>0. \end{cases}$$

We use again the L^2-theory of the Hilbert transform. In particular, using the Fourier transform we have

$$\begin{aligned}\langle HM_{\beta n}T_{\alpha m}\gamma, M_{\beta l}T_{\alpha k}\gamma\rangle &= \langle (HM_{\beta n}T_{\alpha m}\gamma)^{\wedge}, (M_{\beta l}T_{\alpha k}\gamma)^{\wedge}\rangle \\ &= -i\langle S\cdot (M_{\beta n}T_{\alpha m}\gamma)^{\wedge}, (M_{\beta l}T_{\alpha k}\gamma)^{\wedge}\rangle \\ &= -ie^{2\pi i\alpha\beta(kl-mn)}\,\langle S\cdot M_{-\alpha m}T_{\beta n}\hat{\gamma}, M_{-\alpha k}T_{\beta l}\hat{\gamma}\rangle \\ &= -ie^{2\pi i\alpha\beta k(n-l)}V_{\hat{\gamma}}((T_{-\beta n}S)\cdot\hat{\gamma})(\beta(l-n),\alpha(m-k)). \quad (69)\end{aligned}$$

With that choice of γ it is easy to see that for all $n\in\mathbb{Z}$,

$$T_{-\beta n}S\cdot\hat{\gamma} = \pm\hat{\gamma}. \quad (70)$$

Hence, (69) becomes

$$\langle HM_{\beta n}T_{\alpha m}\gamma, M_{\beta l}T_{\alpha k}\gamma\rangle = \pm ie^{2\pi i\alpha\beta k(n-l)}V_{\hat{\gamma}}\hat{\gamma}(\beta(l-n),\alpha(m-k)). \quad (71)$$

By putting all the above together into (65) we have

$$\langle Hf, g\rangle = \pm i\sum_{m,n\in\mathbb{Z}}\sum_{k,l\in\mathbb{Z}} e^{2\pi i\alpha\beta k(n-l)}V_{\hat{\gamma}}\hat{\gamma}(\beta(l-n),\alpha(m-k))\,\langle f, M_{\beta n}T_{\alpha m}\phi\rangle\,\overline{\langle g, M_{\beta l}T_{\alpha k}\phi\rangle}. \quad (72)$$

Taking the magnitude of both sides then yields the desired result. □

We are now ready to prove the boundedness of the Hilbert transform on all the modulation spaces $M^{p,q}$ with $1<p,q<\infty$.

Theorem 5.2.3. *Let $1<p,q<\infty$, then the Hilbert transform H extends to a bounded linear operator on $M^{p,q}$. In particular, for any $f\in M^{p,q}$ we have the following estimate:*

$$\|Hf\|_{M^{p,q}} \le C\,\|f\|_{M^{p,q}},$$

for some positive constant C independent of f.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R})$, then, by Proposition 68, we have that

$$\begin{aligned}
|\langle Hf, g\rangle| &\leq \sum_{m,n\in\mathbb{Z}} |\langle f, M_{\beta n}T_{\alpha m}\phi\rangle| \sum_{k,l\in\mathbb{Z}} A_{\alpha,\beta,\gamma}(m-k, l-n)\, |\langle g, M_{\beta l}T_{\alpha k}\phi\rangle| \\
&= \sum_{n,m} |\langle f, M_{\beta n}T_{\alpha m}\phi\rangle| \left(A_{\alpha,\beta,\gamma} * |\langle g, M_{\beta \cdot}T_{\alpha \cdot}\phi\rangle|(m,n)\right) \\
&\leq \|\langle f, M_{\beta \cdot}T_{\alpha \cdot}\phi\rangle\|_{\ell^{p,q}} \, \|A_{\alpha,\beta,\gamma} * |\langle g, M_{\beta \cdot}T_{\alpha \cdot}\phi\rangle|\|_{\ell^{p',q'}} \\
&\leq \|A_{\alpha,\beta,\gamma}\|_{\ell^1} \, \|\langle f, M_{\beta \cdot}T_{\alpha \cdot}\phi\rangle\|_{\ell^{p,q}} \, \|\langle g, M_{\beta \cdot}T_{\alpha \cdot}\phi\rangle\|_{\ell^{p',q'}} \\
&\leq C\, \|\gamma\|_{M^1} \, \|f\|_{M^{p,q}} \, \|g\|_{M^{p',q'}}.
\end{aligned}$$

We have used Young's inequality to obtain the fourth inequality. By duality we then obtain

$$\|Hf\|_{M^{p,q}} \leq C\, \|\gamma\|_{M^1} \, \|f\|_{M^{p,q}}.$$

The result then follows since $\mathcal{S}(\mathbb{R})$ is a dense subspace of each of the modulation spaces $M^{p,q}$ for $1 < p, q < \infty$. ☐

Remark 5.2.4. The technique that we use to prove the boundedness of H on the modulation spaces is different from the one used in the L^p theory, but relies heavily on the L^2 theory.

We remark that the H cannot be bounded on the Feichtinger algebra M^1. To see this notice that M^1 is a dense subspace of L^2, and that functions in M^1 are continuous. Since M^1 is invariant under the Fourier transform, it is easy to see that for $f \in M^1$,

$$Hf \in M^1 \iff \widehat{Hf} \in M^1.$$

However, $\widehat{Hf} = -S \cdot \hat{f}$, and this function cannot belong to M^1 since it has a discontinuity at the origin.

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