Isoperimetric problems arising in the physics of thin structures and in geometry

Evans Harrell
Georgia Tech
www.math.gatech.edu/~harrell

Seminar
University of Arizona
24 February 2006
Nanoelectronics

- Quantum wires
- Quantum waveguides
- Designer potentials - STM places individual atoms on a surface; quantum dots
- Semi- and non-conducting “threads”
  Simplified mathematical models
An electron near a charged thread

LMP 2006, with Exner and Loss

$$H_{\alpha, \Gamma} = -\Delta - \alpha \delta(x - \Gamma)$$

Fix the length of the thread. What shape binds the electron the least tightly? Conjectured for about 3 years that answer is circle.
Reduction to an isoperimetric problem of classical type.

Is it true that:

\[
\int_0^L |\Gamma(s+u) - \Gamma(s)| \, ds \leq \frac{L^2}{\pi} \sin \frac{\pi u}{L}
\]
Reduction to an isoperimetric problem of classical type.

Birman-Schwinger reduction. A negative eigenvalue of the Hamiltonian corresponds to a fixed point of the Birman-Schwinger operator:

$$\mathcal{R}^\kappa_{\alpha, \Gamma} \phi = \phi, \quad \mathcal{R}^\kappa_{\alpha, \Gamma}(s, s') := \frac{\alpha}{2\pi} K_0\left(\kappa |\Gamma(s) - \Gamma(s')|\right)$$

$K_0$ is the Macdonald function (Bessel function that is the kernel of the resolvent in 2 D).
About Birman-Schwinger

With a factorization due to Birman and Schwinger, an operator $H$ will have eigenvalue $\lambda$ iff the family of operators $B(\lambda)$ has eigenvalue 1.
Consider the negative eigenvalues of 
\(-\Delta + V(x)\) on \(\mathbb{R}^n\), \(V(x)\) minimal, regular (Kato class) and \(\to 0\) at \(\infty\).

Suppose \(u\) is an eigenfunction,
\[ (-\Delta + V(x)) u = \lambda u, \quad \lambda < 0. \]

\[ (-\Delta + |x|) u = -V(x) u \]
\[ u = (-\Delta + |x|)^{-1} V(x) u. \]

Let \(\phi = \sqrt{|x|} \operatorname{sgn} V(x) u\). Then
\[ \phi = \sqrt{|x|} \operatorname{sgn} V(x) (-\Delta H)^{-1} V(x) u \]
\[ =: B_\lambda \phi. \]

Suppose \(V(x) < 0\). Then \(B_\lambda = \sqrt{|x|} (-\Delta H)^{-1} V(x)\) and
\[ \phi = B_\lambda \phi \] (eigenvalue \(\lambda\)).
Now note that if $\lambda$ is in the sense of operators,

\[ 0 > \lambda > \lambda' \implies B_\lambda > B_{\lambda'} \]

Also,

\[ \lim_{\lambda \to -\infty} B_\lambda = 0 \]

\[ \sigma(B_\lambda) \]
Consider now two potentials $V_1, V_2$, and suppose $B_{\lambda}^1 \geq B_{\lambda}^2$ for all $\lambda < 0$. Then $\lambda_k(-A+V_1) \leq \lambda_k(-A+V_2)$ because all the curves of $B_{\lambda}^1$ are higher than those of $B_{\lambda}^2$.

\[
\begin{align*}
\lambda_k \quad & \quad \lambda_k \\
B_{\lambda}^1 \quad & \quad B_{\lambda}^2 \\
& \quad 1
\end{align*}
\]
A positively charged thread can be modeled as
\[ V(x) = -\alpha \delta_p(x) \]

Approximate by honest functions supported in \( \mathbb{E} \): \( \text{dist}(x, \mathbb{E}) \leq \varepsilon,\)

\[ \exists \text{ has a kernel like} \]

\[ -\delta(x) G(\frac{x - y}{h}, 1) \]

This converges to an integral kernel

\[ G(\frac{x - y}{h}, 1) \quad \forall x, y \in \mathbb{R}^n \]

Specifically, with \( k = \sqrt{x} \),

\[ \frac{\alpha}{2\pi} k^0 \left| (k | \Pi(x) - \Pi(s') |) \right| \]
It suffices to show that the largest eigenvalue of $\mathcal{R}_{\alpha,\Gamma}^\kappa$ is uniquely minimized by the circle, i.e.,

\[
\int_0^L \int_0^L K_0(\kappa|\Gamma(s) - \Gamma(s')|) \, ds \, ds' \geq \int_0^L \int_0^L K_0(\kappa|C(s) - C(s')|) \, ds \, ds'
\]

with equality only for the circle.
It suffices to show that the largest eigenvalue of \( \mathcal{R}^{\kappa}_{\alpha, \Gamma} \) is uniquely minimized by the circle, i.e.,

\[
\int_0^L \int_0^L K_0(\kappa|\Gamma(s) - \Gamma(s')|) \, ds \, ds' \geq \int_0^L \int_0^L K_0(\kappa|\mathcal{C}(s) - \mathcal{C}(s')|) \, ds \, ds'
\]

with equality only for the circle. Equivalently, show that

\[
F_\kappa(\Gamma) := \int_0^{L/2} du \int_0^L ds \left[ K_0(\kappa|\Gamma(s+u) - \Gamma(s)|) - K_0 \left( \frac{\kappa L}{\pi} \sin \frac{\pi u}{L} \right) \right]
\]

is positive (0 for the circle).
Since $K_0$ is decreasing and strictly convex, with Jensen’s inequality,

$$\frac{1}{L} F_\kappa(\Gamma) \geq \int_0^{L/2} \left[ K_0 \left( \frac{\kappa}{L} \int_0^L |\Gamma(s+u) - \Gamma(s)| ds \right) - K_0 \left( \frac{\kappa L}{\pi} \sin \frac{\pi u}{L} \right) \right] du,$$

where the inequality is strict unless $\int_0^L |\Gamma(s+u) - \Gamma(s)| ds$ is independent of $s$,

i.e. for the circle. The conjecture has been reduced to:

$$\int_0^L |\Gamma(s+u) - \Gamma(s)| ds \leq \frac{L^2}{\pi} \sin \frac{\pi u}{L}$$
A family of isoperimetric conjectures for $p > 0$:

\[
C_L^p(u) : \quad \int_0^L |\Gamma(s+u) - \Gamma(s)|^p \, ds \leq \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L},
\]

\[
C_{-L}^{-p}(u) : \quad \int_0^L |\Gamma(s+u) - \Gamma(s)|^{-p} \, ds \geq \frac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}},
\]

Right side corresponds to circle.
Proposition 2.1.

\[ C^p_L(u) \text{ implies } C^{p'}_L(u) \text{ if } p > p' > 0. \]

\[ C^p_L(u) \text{ implies } C^{-p}_L(u) \]

First part follows from convexity of \( x \to x^a \) for \( a > 1 \):

\[
\frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L} \geq \int_0^L \left( |\Gamma(s+u) - \Gamma(s)|^{p'} \right)^{p/p'} \, ds \
\geq L \left( \frac{1}{L} \int_0^L |\Gamma(s+u) - \Gamma(s)|^{p'} \, ds \right)^{p/p'} .
\]
Proof when $p = 2$

\[ \Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n e^{ins} \]

\[ c_{-n} = \overline{c_n} . \]

\[ \dot{\Gamma}(s) = i \sum_{0 \neq n \in \mathbb{Z}} n c_n e^{ins} . \]
By assumption, $|\hat{\Gamma}(s)| = 1$, and hence from

$$2\pi = \int_0^{2\pi} |\hat{\Gamma}(s)|^2 \, ds = \int_0^{2\pi} \sum_{0 \neq m \in \mathbb{Z}} \sum_{0 \neq n \in \mathbb{Z}} n m \, c_m^* \cdot c_n \, e^{i(n-m)s} \, ds,$$

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1. \quad (2.5)$$
By assumption, $|\hat{\Gamma}(s)| = 1$, and hence from

$$2\pi = \int_{0}^{2\pi} |\hat{\Gamma}(s)|^2 \, ds = \int_{0}^{2\pi} \sum_{0 \neq m \in \mathbb{Z}} \sum_{0 \neq n \in \mathbb{Z}} nm \, c_m^* \cdot c_n \, e^{i(n-m)s} \, ds,$$

so

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1. \tag{2.5}$$

$$\int_{0}^{2\pi} \left| \sum_{0 \neq n \in \mathbb{Z}} c_n (e^{imu} - 1) e^{ins} \right|^2 \, ds = 8\pi \sum_{0 \neq n \in \mathbb{Z}} |c_n|^2 \left( \sin \frac{nu}{2} \right)^2,$$
Inequality equivalent to

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 \left( \frac{\sin \frac{n u}{2}}{n \sin \frac{u}{2}} \right)^2 \leq 1.$$
It is therefore sufficient to prove that

$$|\sin nx| \leq n \sin x$$

Inductive argument based on

$$(n + 1) \sin x \mp \sin(n + 1)x = n \sin x \mp \sin nx \cos x + \sin x (1 \mp \cos nx)$$
What about $p > 2$?

Funny you should ask....
What about $p > 2$?

Funny you should ask….

The conjecture is false for $p = \infty$. The family of maximizing curves for $\|\Gamma(s+u) - \Gamma(s)\|_\infty$ consists of all curves that contain a line segment of length $> s$. 
What about $p > 2$?

Funny you should ask….

The conjecture is false for $p = \infty$. The family of maximizing curves for $\|\Gamma(s+u) - \Gamma(s)\|_\infty$ consists of all curves that contain a line segment of length $> s$.

At what critical value of $p$ does the circle stop being the maximizer?
What about $p > 2$?

At what critical value of $p$ does the circle stop being the maximizer?

This problem is open. We calculated $\|\Gamma(s+u) - \Gamma(s)\|_p$ for some examples:

Two straight line segments of length $\pi$:

$\|\Gamma(s+u) - \Gamma(s)\|_p^p = 2^{p+2}(\pi/2)^{p+1}/(p+1)$.

Better than the circle for $p > 3.15296\ldots$
What about $p > 2$?

Examples that are more like the circle are not better than the circle until higher $p$:

Stadium, small straight segments  $p > 4.27898...$
What about $p > 2$?

Examples that are more like the circle are not better than the circle until higher $p$:

Stadium, small straight segments $p > 4.27898\ldots$

Polygon with many sides, $p > 6$
What about $p > 2$?

Examples that are more like the circle are not better than the circle until higher $p$:

Stadium, small straight segments $p > 4.27898…$

Polygon with many sides, $p > 6$

Polygon with rounded edges, similar.
Circle is local maximizer for all $p < \infty$

Let $\Gamma(\gamma, s)$ be a closed curve in the complex plane parametrized by arc length $s$, of the form $(1 - \gamma)e^{is} + \Theta(\gamma, s)$, where $\gamma \geq 0$. Suppose that $\Theta$ is smooth (say, $C^2$ in $\gamma$ and $s$), and that for each $\gamma$, $\Theta(\gamma, s)$ is orthogonal to $e^{is}$. Then $\Gamma(0, s)$ is a circle of radius 1, and for any $u$, $0 < u < 2\pi$,

$$\left. \frac{\partial I(\Gamma(\gamma), p, u)}{\partial \gamma} \right|_{\gamma=0} < 0.$$
On a (hyper) surface, what object is most like the Laplacian?

(\Delta = \text{the good old flat scalar Laplacian of Laplace})
Answer #1 (Beltrami’s answer):

Consider only tangential variations.
Answer #1 (Beltrami’s answer):

Consider only tangential variations.

Difficulty:

- The Laplace-Beltrami operator is an intrinsic object, and as such is unaware that the surface is immersed!
Answer #2

The nanoelectronics answer


\[- \Delta_{LB} + q,\]

\[q(x) = \frac{1}{4} \left( \sum_{j=1}^{d} \kappa_j \right)^2 - \frac{1}{2} \sum_{j=1}^{d} \kappa_j^2\]

\[d=1, \quad q = -\kappa^2/4 \leq 0 \quad d=2, \quad q = - (\kappa_1 - \kappa_2)^2/4 \leq 0\]
Some other answers

- In other physical situations, such as reaction-diffusion, $q(x)$ may be other quadratic expressions in the curvature, usually $q(x) \leq 0$.

- The conformal answer: $q(x)$ is a multiple of the scalar curvature.
Heisenberg's Answer
(if he had thought about it)

\[ q(x) = \frac{1}{4} \left( \sum_{j=1}^{d} \kappa_j \right)^2 \]
Heisenberg's Answer
(if he had thought about it)

\[ q(x) = \frac{1}{4} \left( \sum_{j=1}^{d} \kappa_j \right)^2. \]

Note: \( q(x) \geq 0 \) !
Some more loopy problems

\[ H(g) := -\frac{d^2}{ds^2} + g \kappa^2. \]
The isoperimetric theorems for
\[- \nabla^2 + q(\kappa)\]

I. One dimension

\[- \frac{q}{ds^2} + g \kappa^2\]
- curve.

A. \( R \) infinitely long, asymptotically straight

1. \( g < 0\)
   \( \lambda_1 < 0 \) unless \( R \) is a line

2. \( \lambda_1 \) uniquely maximized by
   Ducloux-Exner

B. \( \lambda_1 \) closed, say \( |R| = 1 \)

(i) \( g < 0\)
   \( \lambda_1 \) maximized by \( 0 \)
   Ducloux-Exner

(ii) \( g = -1\)
   \( \lambda_2 \) uniquely maximized by \( 0 \)
   Hamweil-Loss

(iii) \( 0 \leq g \leq \frac{1}{4}\)
   \( \lambda_1 \) uniquely minimized by \( 0 \)
   Exner-Hamweil Loss
Minimality when $g \leq 1/4$.

Proof. a) Assume first that $0 < g < 1/4$. The minimal value of $\lambda_1$, which we denote by $\lambda_*$, is

$$\inf_{\kappa} \inf_{\zeta} \int \left( \left( \frac{d\zeta}{ds} \right)^2 + g \kappa^2 \zeta^2 \right) ds,$$
Because the quantity in question is an iterated infimum, it may be calculated in the other order. By Cauchy-Schwarz’s inequality

$$2\pi = \int \frac{\kappa}{\zeta} ds \leq \left( \int \frac{1}{\zeta^2} ds \right)^{1/2} \left( \int \kappa^2 \zeta^2 ds \right)^{1/2},$$

with equality only if

$$\kappa = \left( 2\pi / \int \frac{1}{\zeta^2} ds \right)^{\frac{1}{2}} \frac{1}{\zeta^2}.$$
A non linear functional

\[ E(\zeta) := \int \left( \frac{d\zeta}{ds} \right)^2 ds + \frac{4\pi^2 g}{\int \left( \frac{1}{\zeta^2} \right) ds}. \]
A non linear functional

\[ E(\zeta) := \int \left( \frac{d\zeta}{ds} \right)^2 ds + \frac{4\pi^2 g}{\int \left( \frac{1}{\zeta^2} \right) ds}. \]

\[ \lambda_* \leq E(\zeta) = 4g\pi^2 < \pi^2 \text{ for } g < 1/4. \]
Lemma 5: If \( E(\zeta) \leq \pi^2 \) for a positive test function \( \zeta \) normalized in \( L^2 \), then

\[
\inf_s (\zeta(s)) > 1 - \frac{\sqrt{E(\zeta)}}{\pi}.
\]

Proof of Lemma 5.

\[
E(\zeta) > \int_0^1 (\zeta')^2 \, ds = \int_0^1 (\zeta - \zeta_{\text{min}})^2 \, ds \geq \pi^2 \int_0^1 (\zeta - \zeta_{\text{min}})^2 \, ds,
\]
Lemma 5: If $E(\zeta) \leq \pi^2$ for a positive test function $\zeta$ normalized in $L^2$, then

$$\inf_s (\zeta(s)) > 1 - \frac{\sqrt{E(\zeta)}}{\pi}.$$

Proof of Lemma 5.

$$E(\zeta) > \int_0^1 (\zeta')^2 \, ds = \int_0^1 (\zeta - \zeta_{\min})^2 \, ds \geq \pi^2 \int_0^1 (\zeta - \zeta_{\min})^2 \, ds,$$
Minimizer therefore exists. Its Euler equation is

\[-\zeta'' + M \frac{1}{\zeta^3} = C \zeta,\]

\[M = \frac{4\pi^2 g}{\left(\int_0^1 \frac{1}{\zeta^2} \, ds\right)^2}\]
Solution of Euler equation of the form:

\[ \zeta^2 = 1 + \sqrt{1 - \frac{M}{\lambda*}} \cos \left(2\sqrt{\lambda*}(s - s_0)\right) \]

Nonconstant solution of this form excluded because \( \lambda* < \pi^2 \).
The isoperimetric theorems for
- $\Delta^2 + q(\kappa)$

C. Open.

$\lambda_2, g \neq 1, g < 0.$

$\lambda_1, \frac{1}{4} \leq g \leq 1$

Freitas, Non-embedded problem bifurcates at $\frac{1}{4}$

Benjumia-Loss, family of curves with some $\lambda_1$ at 1

Linde, $g = 1$, lower bound under additional assumptions
Progress on $-\frac{d^2}{ds^2} + g \kappa^2$

- Benguria-Loss, *Contemp. Math.* 2004
  - Connection to Lieb-Thirring in one-D
Progress on $-d^2/ds^2 + g \kappa^2$

- Benguria-Loss, *Contemp. Math.* 2004
  - Connection to Lieb-Thirring in one-D
  - Family of curves with same $\lambda_0$ as circle when $g=1$
Progress on \(-d^2/ds^2 + g \kappa^2\)

- Benguria-Loss, *Contemp. Math.* 2004
  - Connection to Lieb-Thirring in one-D
  - Family of curves with same \(\lambda_0\) as circle when \(g=1\)
    - \(\lambda_0 > 1/2\).
Progress on \(-d^2/ds^2 + g \kappa^2\)

- Linde Proc AMS 2005
- \(\lambda_0 > 0.6085\) (convex, etc.)
II. Two Dimensions

A. \( g + \xi = g, k_1, k_2 \) (Gauss curvature), \( \text{genus}(\mathbb{R}) = 0, \mathbb{R}^2 \).

(i) Hersch 1970, \( g = 0 \), \( \lambda, \text{uniquely} = 0 \)

\[ d = 2 \quad \lambda, \text{uniquely maximized by} \quad g^2 < R^2 \quad \bigcirc \]

(ii) Harrell 1976, any \( g \), \( \lambda, g \) both uniquely maximized by \( \bigcirc \)

* Certain other potentials, \( g (k_1^2 + k_2^2) g \leq 0 \).

Open - other genera

* Special facts in 2-D about conformal equivalence.
II. Two dimensions

A. \( g+h^2 = g^2, k_1, k_2 \) (Gauss curvature), \( \text{genus}(\mathbb{R})=0, |X|=1 \).

False in high dim.

1. \( \text{Heisen 1970, } g=0 \), \( \lambda_1 \text{ triviality } = 0 \)
   \( d=2 \), \( \lambda_2 \) uniquely maximized by \( g^2 \subset \mathbb{R}^2 \) \( \bigcirc \)

2. \( \text{Harnell 1996, any } g \), \( \lambda_2 \) both uniquely maximized by \( \bigcirc \)
   + certain other potentials, \( g (K_1^2 + K_2^2) g < 0 \).

Open - other genera

* Special facts in 2-D about conformal equivalence.
III Two or more dimensions.

A $\mathbb{R}$-hypersurface of codimension 1

$$-\nabla^2 - \frac{1}{\dim} (\sum K_e)^2$$

$\lambda_2$ uniquely maximized by sphere

(Hamilton-Loss '98).

$\Rightarrow$ Same for $\varphi(t) = -\sum K_e^2$

B) $\Sigma$-embedded in $\mathbb{R}^{m+1}$, $H^{m+1}$, $\mathbb{R}^n$

El Soufi-Îl'in.

Actually show $\lambda_2 (-\nabla^2 + \varphi(t)) \leq \frac{1}{\dim} \int \sum K_e^2 + \varphi(t)$. 

Universal Bounds using Commutators

• A “sum rule” identity (Harrell-Stubbe, 1997):

\[
1 = \frac{4}{d} \sum_{k: \lambda_k \neq \lambda_j} \frac{|\langle u_k, p u_j \rangle|^2}{\lambda_k - \lambda_j}
\]

Here, H is any Schrödinger operator, p is the gradient (times -i if you are a physicist and you use atomic units)
Commutators:  \([A, B] := AB - BA\)

3a. The equations of space curves are commutators:

\[
\left[ \frac{d}{ds}, x \right] = t
\]

\[
\left[ \frac{d}{ds}, t \right] = \kappa n
\]

Note: curvature is defined by a second commutator
The Serret-Frenet equations as commutator relations:

\[ [H, X_m] = -\frac{d^2 X_m}{ds^2} - 2 \frac{d X_m}{ds} \frac{d}{ds} = -\kappa n_m - 2 t_m \frac{d}{ds}, \quad (2.2) \]

\[ [X_m [H, X_m]] = 2 t_m^2. \quad (2.3) \]
Lemma. Let $M$ be a smooth curve in $\mathbb{R}^d$, $d = 2$ or 3. Then for

$$H = - \frac{d^2}{ds^2} + V(s) \quad \text{and} \quad \varphi \in W_0^1(M),$$

$$\sum_{m=0}^{d} \| [H, X_m] \varphi \|^2 = 4 \int_M \left( \left| \frac{d\varphi}{ds} \right|^2 + \frac{\kappa^2}{4} |\varphi|^2 \right) ds.$$
Proposition 2.1 Let $M$ be a smooth curve in $\mathbb{R}^\nu$, $\nu = 2$ or 3. Then for

$$H = -\frac{d^2}{ds^2} + V(s) \quad \text{and} \quad \varphi \in W_0^1(M),$$

$$\sum_{m=0}^{d} \|[H, X_m] \varphi\|^2 = 4 \int_M \left( \left|\frac{d\varphi}{ds}\right|^2 + \frac{\kappa^2}{4} |\varphi|^2 \right) ds.$$

Proof. By closure it may be assumed that $\varphi \in C_0^\infty(M)$. Apply (2.2) to $\varphi$ and square the result, to obtain

$$4 \left( t_m^2 \left( \frac{d\varphi}{ds} \right)^2 + \frac{1}{4} \kappa^2 n_m^2 \varphi^2 + \frac{1}{2} \kappa n_m t_m \varphi \frac{d\varphi}{ds} \right).$$

Sum on $m$ and integrate. \hspace{1cm} QED
Interpretation:

Algebraically, for quantum mechanics on a wire, the natural $H_0$ is not $p^2$, but rather $H_{1/4} := p^2 + \kappa^2/4$. 
Corollary 2.2 Let $M$ be as in Proposition 2.1 and suppose that $H$ is a Schrödinger Hamiltonian with a bounded measurable potential $V(s)$. Then

$$
\Gamma \leq 4 \int_M \left( \left( \frac{du_1}{ds} \right)^2 + \frac{\kappa^2}{4} u_1^2 \right) ds.
$$

(2.5)
Corollary 2.2 Let $M$ be as in Proposition 2.1 and suppose that $H$ is a Schrödinger Hamiltonian with a bounded measurable potential $V(s)$. Then

$$\Gamma \leq 4 \int_M \left( \left( \frac{du_1}{ds} \right)^2 + \frac{\kappa^2}{4} u_1^2 \right) ds. \quad (2.5)$$

That is, the gap for any $H$ is controlled by an expectation value of $H_{1/4}$. 
Corollary 2.2 Let $M$ be as in Proposition 2.1 and suppose that $H$ is a Schrödinger Hamiltonian with a bounded measurable potential $V(s)$. Then

$$
\Gamma \leq 4 \int_M \left( \left( \frac{du_1}{ds} \right)^2 + \frac{\kappa^2}{4} u_1^2 \right) ds. \tag{2.5}
$$

Furthermore, if $H$ is of the form

$$
H_g := -\frac{d^2}{ds^2} + g\kappa^2,
$$

then

$$
\Gamma \leq \max \left( 4, \frac{1}{g} \right) \lambda_1. \tag{2.6}
$$

Equivalently, the universal ratio bound

$$
\frac{\lambda_2}{\lambda_1} \leq \max \left( 5, 1 + \frac{1}{g} \right)
$$

holds.
Bound is sharp for the circle:

\[ \frac{\lambda_2}{\lambda_1} = \frac{4\pi^2 (1 + g)}{4\pi^2 g} = 1 + \frac{1}{g}. \]
Gap bounds for (hyper) surfaces

Let $M$ be a $d$-dimensional manifold immersed in $\mathbb{R}^{d+1}$.

**Theorem 3.1** Let $H$ be a Schrödinger operator on $M$ with a bounded potential, i.e.,

$$H = -\Delta + V,$$  \hspace{1cm} (3.1)

$$\Gamma(H) \leq \frac{1}{d} \int_M \left( 4|\nabla_{\parallel} u_1|^2 + h^2 u_1^2 \right) dVol$$

$$= \frac{4}{d} \left\langle u_1, \left( -\Delta + \frac{h^2}{4} \right) u_1 \right\rangle.$$

Here $h$ is the sum of the principal curvatures.
Corollary 3.2 Let $H$ be as in (3.1) and define $\delta := \sup_M \left( \frac{h^2}{4} - V \right)$. Then

$$\Gamma(H) \leq \frac{4}{d} (\lambda_1 + \delta).$$
Bound is sharp for the sphere:

\[ \lambda_1 = gd^2, \quad \lambda_2 = gd^2 + d \]

\[ d = \lambda_2 - \lambda_1 \leq \left( \frac{gd^2}{gd} \right) = d. \]
Spinorial Canonical Commutation

\[ P = \sum_{j=1}^{d} \left( t_j \frac{\partial}{\partial s_j} \pm \frac{1}{2} \kappa_j n \right) \]  \hspace{1cm} (4.1)

and for a dense set of functions \( \varphi \),

\[ \| P \varphi \|^2 = \langle \varphi, H_{1/4} \varphi \rangle. \]  \hspace{1cm} (4.2)
Spinorial Canonical Commutation

\[ P = \sum_{j=1}^{d} \left( t_j \frac{\partial}{\partial s_j} \pm \frac{1}{2} \kappa_j n \right) \quad (4.1) \]

and for a dense set of functions \( \varphi \),

\[ \| P \varphi \|^2 = \langle \varphi, H_{1/4} \varphi \rangle . \quad (4.2) \]

Thus \( P \) plays the rôle of a momentum operator, with which there is a version of canonical commutation (cf. (1.9)) as follows. Defining a variant commutator bracket for operators \( L^2(M) \rightarrow \mathbb{R}^{d+1} \otimes L^2(M) \) by \([A; B] := A \cdot B - B \cdot A\), a calculation shows that \([P; X_k e_k] = \sum_{j=1}^{d} t_j \cdot \frac{\partial X_k e_k}{\partial s_j} = 1 \) (identity operator), and by averaging on \( k \),

\[ 1 = \frac{1}{d} [P; X] \quad (4.3) \]

which is a coordinate-independent formula.
Proposition 4.1 Let $H$ be as in (3.1), with eigenvalues $\{\lambda_k\}$ and normalized eigenfunctions $\{u_k\}$. Then

$$1 = \frac{4}{d} \sum_{\substack{k \neq j \atop \lambda_k \neq \lambda_j}} \frac{|\langle u_k, Pu_j \rangle|^2}{\lambda_k - \lambda_j}. \quad (4.4)$$
Corollaries of sum rules

• Sharp universal bounds for all gaps

• Some estimates of partition function
  \[ Z(t) = \sum \exp(-t \lambda_k) \]
Speculations and open problems

• Can one obtain/improve Lieb-Thirring bounds as a consequence of sum rules?
• Full understanding of spectrum of $H_g$.
  What spectral data needed to determine the curve?
  What is the bifurcation value for the minimizer of $\lambda_1$?
• Physical understanding of $H_g$ and of the spinorial operators it is related to.
Corollary 4.4  \( b) \) For \( H_g \) be of the form (1.10) on a smooth, compact submanifold. Then

\[
[
\lambda_n, \lambda_{n+1}\right) \subseteq \left[
\left(1 + \frac{2\sigma}{d}\right) \lambda_n - \sqrt{D_n}, \left(1 + \frac{2\sigma}{d}\right) \lambda_n + \sqrt{D_n}\right],
\]

with

\[
D_n := \left(\left(1 + \frac{2\sigma}{d}\right) \lambda_n\right)^2 - \left(1 + \frac{4\sigma}{d}\right) \lambda_n^2.
\]

This bound is sharp for every non-zero eigenvalue gap of \( H_{\frac{1}{4}} \) on the sphere.
Partition function

\[ Z(t) := \text{tr}(\exp(-tH)). \]
Partition function

\[ Z(t) \leq \left( \frac{2t}{d} \right) \sum_j \left( \exp \left( -t \lambda_j \right) \right) \| Pu_j \|^2, \]
which implies

Corollary 4.5  

a) Let $H$ be as (3.1), with $M$ a compact, smooth submanifold. Then $t^\frac{d}{2} \exp(-\delta t) Z(t)$ is a nondecreasing function;

b) For $H_g$ be of the form (1.10) on a smooth, compact submanifold $M$, $t^\frac{d}{2\sigma} Z(t)$ is a nondecreasing function.